1. (a)

$$\begin{aligned}\int\int_D (1/x)\,dA &= \int_1^e \int_{y^2}^{y^4} \frac{1}{x}\,dx\,dy \\ &= \int_1^e \ln(y^4)-\ln(y^2)\,dy \\ &= 2\int_1^e \ln y\,dy \\ &= 2\,(y\ln y - y)\big|_{y=1}^{e} \\ &= 2.\end{aligned}$$

   (b) The region we are integrating over is $\{(x,y)\in\mathbb{R}^2 \mid 0\le y\le 1, 3y\le x\le 3\}$. This is the same as $\{(x,y)\in\mathbb{R}^2 \mid 0\le y\le x/3, 0\le x\le 3\}$. Therefore,

$$\begin{aligned}\int_0^1\int_{3y}^3 e^{x^2}\,dx\,dy &= \int_0^3\int_0^{x/3} e^{x^2}\,dy\,dx \\ &= \frac{1}{6}\int_0^3 2xe^{x^2}\,dx \\ &= \frac{1}{6}\int_0^9 e^u\,du && \text{(with } u=x^2\text{)} \\ &= \frac{1}{6}(e^9-1).\end{aligned}$$

2. (a) We have that $x=\xi$, $y=e^{\xi+\eta}$, $z=(1+\xi^2+\eta^2+\zeta^2)^{-\frac{1}{2}}$, and $w=\xi-\eta$. So,

$$\begin{aligned}\frac{\partial u}{\partial \xi} &= \frac{\partial u}{\partial x}\frac{\partial x}{\partial \xi}+\frac{\partial u}{\partial y}\frac{\partial y}{\partial \xi}+\frac{\partial u}{\partial z}\frac{\partial z}{\partial \xi}+\frac{\partial u}{\partial w}\frac{\partial w}{\partial \xi} \\ &= \frac{\partial u}{\partial x}+e^{\xi+\eta}\frac{\partial u}{\partial y}-\xi(1+\xi^2+\eta^2+\zeta^2)^{-\frac{3}{2}}\frac{\partial u}{\partial z}+\frac{\partial u}{\partial w}.\end{aligned}$$

   (b) Since x, y, and z all don't depend on w, so there will be a zero column in the Jacobian matrix. Therefore this matrix will have zero determinant. So,

$$\frac{\partial(x,y,z)}{\partial(u,v,w)}=0.$$

3. (a) See the book.

   (b) We can show this with Lagrange multipliers. First,

$$\nabla A(x,y,z) = -\frac{1}{2\sqrt{s(s-a)(s-b)(s-c)}}\langle (s-y)(s-z),(s-x)(s-z),(s-x)(s-y)\rangle.$$

   Our constraint is $g(x,y,z)=x+y+z=2s$. So, $\nabla g(x,y,z)=\langle 1,1,1\rangle$. These two vectors are scalar multiples of each other if and only if

$$(s-y)(s-z)=(s-x)(s-z)=(s-x)(s-y).$$

   Since any side length is only equal to s in the case of a degenerate triangle (which does not have maximal area), this implies that $x=y=z$. Therefore, the triangle with maximal area given a fixed perimeter is equilateral.

4. The given function is $f(x,y)g(z)$, where $f(x,y) = x^2 + 2xy + 3y^2 - 5$ and $g(z) = \sin^2 z$. First we need to find the absolute maxima and minima of f on D. $\nabla f(x,y) = \langle 2x + 2y, 2x + 6y \rangle$, which is only zero at $(0,0)$. $f(-1,y) = 3y^2 - 2y - 4$, which has a critical point at $(-1, 1/3)$. $f(x,1) = x^2 + 2x - 2$, which has a critical point at $(-1,1)$. $f(x, x-1) = x^2 + 2x^2 - 2x + 3x^2 - 6x + 3 - 5 = 6x^2 - 8x - 2$, which has a critical point at $(2/3, -1/3)$. So, the global minimum and maximum of f on D is among

$$\begin{aligned} f(0,0) &= -5 \\ f(-1,1/3) &= -\frac{13}{3} \\ f(-1,1) &= -3 \\ f(2/3,-1/3) &= -\frac{14}{3} \\ f(2,1) &= 2 \\ f(-1,-2) &= 12. \end{aligned}$$

So, the global maximum of f is 12 at $(-1,-2)$ and the minimum is -5 at $(0,0)$.

Since $g(z) \geq 0$ for all z, the absolute maxima of $f(x,y)g(z)$ are at $\left(-1,-2,\frac{\pi}{2}+n\pi\right)$ for all $n \in \mathbb{Z}$, and the absolute minima are at $\left(0,0,\frac{\pi}{2}+n\pi\right)$ for all $n \in \mathbb{Z}$.

5. (a) See the book.

   (b) Since $\frac{\partial z}{\partial x} = 1$ and $\frac{\partial z}{\partial y} = 2y$, the surface area is

$$\begin{aligned} \int_0^b \int_0^{ay} \sqrt{2+4y^2}\, dx\, dy &= \frac{a}{8} \int_0^b 8y\sqrt{2+4y^2}\, dy \\ &= \frac{a}{8} \int_2^{2+4b^2} \sqrt{u}\, du \qquad \text{(with } u = 2+4y^2\text{)} \\ &= \frac{a}{12}\left[(2+4b^2)^{\frac{3}{2}} - 2^{\frac{3}{2}}\right]. \end{aligned}$$

   (c)

1. (a) The region D is $\{(x,y) \in \mathbb{R}^2 \mid 0 \leq y \leq 4, y/2 \leq x \leq 6-y\}$. So,

$$\begin{aligned} \iint_D e^x\, dA &= \int_0^4 \int_{y/2}^{6-y} e^x\, dx\, dy \\ &= \int_0^4 e^{6-y} - e^{y/2}\, dy \\ &= \left(-e^{6-y} - 2e^{y/2}\right)\Big|_{y=0}^4 \\ &= e^6 + 2 - e^2 - 2e^2 \\ &= e^6 - 3e^2 + 2. \end{aligned}$$

   (b) The region we are integrating over is $\{(x,y) \in \mathbb{R}^2 \mid 0 \leq x \leq 1, x^2 \leq y \leq 1\}$ which is the same as

$\{(x,y) \in \mathbb{R}^2 \mid 0 \le x \le \sqrt{y}, 0 \le y \le 1\}$. Therefore

$$\begin{aligned}
\int_0^1 \int_{x^2}^1 x^3 \sin(y^3)\,dy\,dx &= \int_0^1 \int_0^{\sqrt{y}} x^3 \sin(y^3)\,dx\,dy \\
&= \frac{1}{12}\int_0^1 3y^2 \sin(y^3)\,dy \\
&= \frac{1}{12}\int_0^1 \sin u\,du && \text{(with } u = y^3\text{)} \\
&= \frac{1}{12}(\cos 1 - \cos 0) \\
&= \frac{\cos 1 - 1}{12}.
\end{aligned}$$

2. See the book.

3.

$$\begin{aligned}
\frac{\partial(x,y,z)}{\partial(u,v,w)} &= \det\begin{pmatrix} 1 & 2v & 1 \\ 0 & 1 & 4w^3 \\ 1 & 0 & -1 \end{pmatrix} \\
&= (8vw^3 - 1) - (1) \\
&= 8vw^3 - 2.
\end{aligned}$$

4. (a) See the second derivative test in the book.

   (b) First, we want to find critical points on the interior of the region. So, $\nabla f(x,y) = \langle 2x + x^2 - y^2, 2y - 2xy\rangle$. If $2y - 2xy = 0$, then either $y = 0$ or $x = 1$. If $y = 0$, then the condition that $2x + x^2 = 0$ tell us that x is either 0 or -2. If $x = 1$, then the condition $3 - y^2 = 0$ tells us that y is $\sqrt{3}$ or $-\sqrt{3}$. So, the four critical points are $(1, \sqrt{3})$, $(1, \sqrt{3})$, $(0,0)$, $(0,-2)$. However, we need only consider critical points inside the region, so the only critical point of any concern is $(0,0)$, and $f(0,0) = 0$.

   We can use Lagrange multipliers to find the minimum and maximum values of f along the boundary of the region. For this, our constraint function will be $g(x,y) = x^2 + y^2$, so that $\nabla g(x,y) = \langle 2x, 2y\rangle$. If $2y(1-x) = 2\lambda y$, then either $\lambda = 1 - x$ or $y = 0$. If $\lambda = 1 - x$, then from the first coordinate we have $2x + x^2 - y^2 = 2x - 2x^2$, or $3x^2 - y^2 = 0$. Since $x^2 + y^2 = 1$ on the boundary, this means $0 = 4x^2 - 1 = (2x+1)(2x-1)$. If $y = 0$, then x is either 1 or -1.

$$\begin{aligned}
f(1,0) &= \frac{4}{3} \\
f(-1,0) &= \frac{2}{3} \\
f\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right) &= \frac{2}{3} \\
f\left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right) &= \frac{4}{3} \\
f\left(\frac{1}{2}, -\frac{\sqrt{3}}{2}\right) &= \frac{2}{3} \\
f\left(-\frac{1}{2}, -\frac{\sqrt{3}}{2}\right) &= \frac{4}{3}
\end{aligned}$$

So, the absolute minimum of f is 0, and the absolute maximum is $\frac{4}{3}$.

5.

$$\begin{aligned} V &= \int_0^1 \int_x^1 x + y^2 \; dy \; dx \\ &= \int_0^1 x - x^2 + \frac{1}{3} - \frac{x^3}{3} \; dx \\ &= \frac{1}{2} - \frac{1}{3} + \frac{1}{3} - \frac{1}{12} \\ &= \frac{5}{12}. \end{aligned}$$

Since $\frac{\partial z}{\partial x} = 1$ and $\frac{\partial z}{\partial y} = 2y$, the surface area of the top surface is

$$\begin{aligned} \int_0^1 \int_0^y \sqrt{2 + 4y^2} \; dx \; dy &= \frac{1}{8} \int_0^1 8y\sqrt{2 + 4y^2} \; dy \\ &= \frac{1}{8} \int_2^6 \sqrt{u} \; du \qquad \text{(with } u = 2 + 4y^2\text{)} \\ &= \frac{1}{12}(6\sqrt{6} - 2\sqrt{2}). \end{aligned}$$

The areas of the other surfaces can be found using single variable methods.