Midterm 2 Review

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1 Partial Derivatives

1.1 Things to Know

- Definition of the limit of a multivariable function.
- Definitions of partial derivatives.
- Clairaut's Theorem: Suppose f is defined on a disk D that contains the point (a, b). If the functions f_{xy} and f_{yx} are both continuous on D, then $f_{xy}(a, b) = f_{yx}(a, b)$.
- Formula for tangent plane.
- Definition of differentiability of a multivariable function.
- If the partial derivatives f_x and f_y exist near (a, b) and are continuous at (a, b), then f is differentiable at (a, b).
- If $z = f(x, y)$, $x = g(t)$, and $y = h(t)$, then

$$\frac{dz}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}.$$

- If $z = f(x, y)$, $x = g(s, t)$, and $y = h(s, t)$, then

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s}$$

  and

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t}.$$

- If $u = f(x_1, \ldots, x_n)$ and each x_i is a differentiable function of $t_1, \ldots, t_m$, then

$$\frac{\partial u}{\partial t_i} = \sum_{j=1}^{n} \frac{\partial u}{\partial x_j}\frac{\partial x_j}{\partial t_i}.$$

- Let f be a function of x and y. The gradient of f at (a, b), $\nabla f(a, b)$, is defined to be the vector $\langle f_x(a, b), f_y(a, b)\rangle$. The definition is analogous for functions with more than two variables. The gradient vector points in the direction of steepest ascent for f. The vector $\nabla f(a, b)$ is always perpendicular to the level curve passing through (a, b).

- The directional derivative of a function of two variables is the derivative of the one variable function you get by taking a slice of the two variable function along a certain direction. It can be computed by taking the dot product of a unit vector for the direction you're looking at and the gradient of the function at that point.
- A function $f(x, y, z)$ of three variables has level surfaces rather than level curves. At any point (a, b, c), the normal vector for the plane tangent to the level surface of f passing through (a, b, c) is given by $\nabla f(a, b, c)$.
- At any local minimum or maximum of any function, the gradient either does not exist or is equal to the zero vector. Points where the gradient vector does not exist or is equal to the zero vector are called critical points, like in 1A.
- Let $f(x, y)$ have continuous partial derivatives on a disk centered at (a, b), and suppose $f_x(a, b) = f_y(a, b) = 0$. Let $D = f_{xx}(a, b)f_{yy}(a, b) - [f_{xy}(a, b)]^2$. Then if $D > 0$ and $f_{xx}(a, b) > 0$, then (a, b) is a local maximum for f. If $D > 0$ and $f_{xx}(a, b) < 0$, then (a, b) is a local minimum. If $D < 0$, then (a, b) is neither a local maximum nor a local minimum.
- Recall from 1A that if you have a continuous function on a closed and bounded set, it must achieve maximum and minimum values. You only need to check the boundary points and the critical points. This applies to functions in any number of variables as well, except that the boundaries and critical points can now be more complicated.
- To maximize $f(x, y, z)$ subject to the constraint $g(x, y, z) = k$, find all values of x, y, z, and λ such that $\nabla f(x, y, z) = \lambda \nabla g(x, y, z)$ and $g(x, y, z) = k$. Then you only need to try these points (ignoring λ).
- To maximize $f(x, y, z)$ subject to the constraints $g(x, y, z) = k$ and $h(x, y, z) = s$, find all values of x, y, z, λ, and μ such that $\nabla f(x, y, z) = \lambda \nabla g(x, y, z) + \mu \nabla h(x, y, z)$, $g(x, y, z) = k$ and $h(x, y, z) = s$. Then you only need to try these points (ignoring λ and μ).

1.2 Problems

1. (Practice Midterm 2 for Midterm 1) Show that the function $u(x, y) = y^{-\frac{1}{2}} e^{-x^2/(4y)}$ solves the following partial differential equation:
$$u_y = u_{xx}.$$
2. (Chapter 14 Review) Evaluate each limit or show that it does not exist:
   (a) $\lim_{(x,y)\to(1,1)} \frac{2xy}{x^2+2y^2}$
   (b) $\lim_{(x,y)\to(0,0)} \frac{2xy}{x^2+2y^2}$
3. (Chapter 14 Review) Find all second partial derivatives of $v(r, s, t) = r\cos(s + 2t)$.
4. (Spring 1993 Prelim[1]) Prove that $\frac{x^2+y^2}{4} \le e^{x+y-2}$ for $x \ge 0$, $y \ge 0$.
5. (Spring 2003 Prelim) Define $f : \mathbb{R}^2 \to \mathbb{R}$ by $f(x, 0) = 0$ and
$$f(x, y) = \left[1 - \cos\left(\frac{x^2}{y}\right)\right] \sqrt{x^2 + y^2}$$
for $y \ne 0$.

[1] A website for the prelim can be found on the math department's web page, by clicking on "Graduate" and then on "Prelim Exam." This site has old exams and some solutions. There is also a book called *Berkeley Problems in Mathematics*, which most GSIs have access to, that has solutions to all prelim problems appearing in this document.

(a) Show that f is continuous at $(0,0)$.

(b) Calculate all the directional derivatives of f at $(0,0)$.

(c) Show that f is not differentiable at $(0,0)$.

6. (a) Find the minimal value of the areas of triangles circumscribing the unit circle in $\mathbb{R}^2$.

   (b) (Fall 1998 Prelim) Find the minimal value of the areas of hexagons circumscribing the unit circle in $\mathbb{R}^2$. (This problem requires calculus of six variables. The previous part only requires three variables.)

7. (Chapter 14 Review) Find the absolute minimum and maximum values of $f(x,y) = e^{-x^2-y^2}(x^2+2y^2)$ on the set $\{(x,y) \in \mathbb{R}^2 \mid x^2+y^2 \leq 4\}$.

8. (Practice Midterm 1) Find the partial derivative $\frac{\partial u}{\partial \xi}$ in terms of the partial derivatives of $u(x,y,z,w)$, where, as a function of (ξ, η, ζ),

$$u = u(\xi, e^{\xi+\eta}, (1+\xi^2+\eta^2+\zeta^2)^{-\frac{1}{2}}, \xi - \eta).$$

9. (Practice Midterm 1) Show that the triangle with a maximal area and with the given length of the perimeter, p, is equilateral. (Hint: Use Heron's formula.)

10. (Practice Midterm 1) Find all the absolute maxima and minima of the function

$$(x^2 + 2xy + 3y^2 - 5)\sin^2 z,$$

on the closed (unbounded) set

$$\{(x,y,z) \in \mathbb{R}^3 \mid (x,y) \in D, z \in \mathbb{R}\},$$

where D is the closed triangular region with vertices at $(-1,1)$, $(2,1)$, and $(-1,-2)$.

11. (Practice Midterm 2) Consider the function

$$f(x,y) = x^2 + \frac{x^3}{3} - xy^2 + y^2.$$

Find the absolute minimum and maximum of f inside the unit disc $x^2 + y^2 \leq 1$.

12. (Chapter 14 Review) Find the points on the surface $xy^2z^3 = 2$ that are closest to the origin.

13. (Chapter 14 Review) A package in the shape of a rectangular box can be mailed by U.S. Parcel Post if the sum of its length and girth (the perimeter of a cross-section perpendicular to the length) is at most 108 in. Find the dimensions of the package with largest volume that can be mailed by Parcel Post.

14. Find the absolute maximum and maximum of the function

$$f(x,y,z) = e^{-z^2}|x| \sin y \cos(x+y)$$

on the set $\{(x,y,z) \in \mathbb{R}^3 \mid x^2+y^2 \leq \pi^2\}$.

2 Multiple Integrals

2.1 Things to Know

- Definitions of double and triple integrals (they are analogous to the single variable case). Same with things like the midpoint rules.
- Fubini's Theorem. (The "iterated integrals exist" means that you get a finite value when you evaluate the iterated integrals with absolute value signs around the function being integrated.)
- How to integrate things over complicated regions.
- If $D = D_1 \cup D_2$ and D_1 and D_2 don't overlap except on their boundaries, then $\int\int_D f(x,y)\,dA = \int\int_{D_1} f(x,y)\,dA + \int\int_{D_2} f(x,y)\,dA$.
- The area of D is $\int\int_D \,dA$.
- If $m \le f(x,y) \le M$ for all $(x,y) \in D$, then $mA(D) \le \int\int_D f(x,y)\,dA \le MA(D)$.
- $dA = r\,dr\,d\theta$.
- To find the mass of an object, m, take the integral of the density function, ρ over the object. The x-coordinate of the center of mass is given by $\overline{x} = \frac{1}{m}\int\int_D x\rho\,dA$. Similarly for the y-coordinate.
- Moment of inertia stuff, expected value stuff.
- Most of the stuff above applies to triple integrals, too.
- The surface area of a surface is given by

$$A(S) = \int\int_D \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\,dA.$$

- $dV = \rho^2 \sin\phi\,d\rho\,d\phi\,d\theta = r\,dr\,d\theta\,dz$.
- The Jacobian of a transformation given by $x = g(u,v)$ and $y = h(u,v)$ is

$$\frac{\partial(x,y)}{\partial(u,v)} = \det\begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{pmatrix}.$$

  It's similar for a three-dimensional transformation.
- Let T be a C^1 transformation mapping R to S. Under some other conditions (look them up),

$$\int\int_R f(x,y)\,dA = \int\int_S f(x(u,v),y(u,v))\left|\frac{\partial(x,y)}{\partial(u,v)}\right|\,du\,dv.$$

  There is a similar formula for three variables.

2.2 Problems

1. (Spring 1978 Prelim) What is the volume enclosed by the ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1?$$

2. (Spring 1978 Prelim) Evaluate

$$\int\int_{\mathcal{A}} e^{-x^2-y^2}\, dx\, dy,$$

where $\mathcal{A} = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq 1\}$.

3. (Spring 1998 Prelim) Given the fact that $\int_{-\infty}^{\infty} e^{-x^2}\, dx = \sqrt{\pi}$, evaluate the integral

$$I = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-(x^2+(y-x)^2+y^2)}\, dx\, dy.$$

4. (Practice Midterm 1) Evaluate $\int\int_D (1/x)\, dA$, where $D = \{(x, y) \mid 1 \leq y \leq e, y^2 \leq x \leq y^4\}$.

5. (Practice Midterm 1) Evaluate $\int_0^1 \int_{3y}^3 e^{x^2}\, dx\, dy$.

6. (Practice Midterm 1) Evaluate the Jacobian

$$\frac{\partial(x, y, z)}{\partial(u, v, w)}$$

of the transformation

$$\begin{aligned} x &= e^u + e^v, \\ y &= e^u - e^v, \\ z &= u + v. \end{aligned}$$

7. (Practice Midterm 1) Find the surface area of the set

$$\{(x, y, z) \in \mathbb{R}^3 \mid z = x + y^2, 0 \leq x \leq ay, 0 \leq y \leq b\}$$

in terms of a and b.

8. (Practice Midterm 2) Evaluate $\int\int_D e^x\, dA$, where D is the triangle with vertices at $(0, 0)$, $(2, 4)$, and $(6, 0)$.

9. (Practice Midterm 2) Evaluate $\int_0^1 \int_{x^2}^1 x^3 \sin\left(y^3\right)\, dy\, dx$.

10. (Practice Midterm 2) Evaluate the Jacobian of the transformation

$$\begin{aligned} x &= u + v^2 + w \\ y &= v + w^4 \\ z &= u - w. \end{aligned}$$

11. (Practice Midterm 2) Consider the following solid in $\mathbb{R}^3$:

$$R = \{(x, y, z) \in \mathbb{R}^3 \mid 0 \leq x \leq 1, x \leq y \leq 1, 0 \leq z \leq x + y^2\}.$$

Find the volume of R and the area of the surface which forms its boundary.

12. (Chapter 15 Review) Use spherical coordinates to evaluate

$$\int_{-2}^{2} \int_{0}^{\sqrt{4-y^2}} \int_{-\sqrt{4-x^2-y^2}}^{\sqrt{4-x^2-y^2}} y^2 \sqrt{x^2 + y^2 + z^2}\, dz\, dx\, dy$$

13. (Chapter 15 Review) Give five other iterated integrals that are equal to

$$\int_0^2 \int_0^{y^3} \int_0^{y^2} f(x, y, z)\, dz\, dx\, dy$$

14. (Chapter 15 Review) Use the transformation $x = u^2$, $y = v^2$, $z = w^2$ to find the volume of the region bounded by the surface $\sqrt{x} + \sqrt{y} + \sqrt{z} = 1$ and the coordinate planes.

15. (Chapter 15 Review) Use the change of variables formula and an appropriate transformation to evaluate $\int\int_R xy\, dA$, where R is the square with vertices $(0, 0)$, $(1, 1)$, $(2, 0)$, and $(1, -1)$.

3 Vector Calculus

3.1 Things to Know

- The terms "vector field" and "conservative vector field."
- Definitions of line integrals.
- Simply-connectedness and the theorems in 16.3.

3.2 Problems

1. (1996 Putnam[2]—VERY TRICKY) Let $(a_1, b_1), (a_2, b_2), \ldots, (a_n, b_n)$ be the vertices of a convex polygon which contains the origin in its interior. Prove that there exist positive real numbers x and y such that both of the following equations hold:

$$a_1 x^{a_1} y^{b_1} + a_2 x^{a_2} y^{b_2} + \cdots + a_n x^{a_n} y^{b_n} = 0$$
$$b_1 x^{a_1} y^{b_1} + b_2 x^{a_2} y^{b_2} + \cdots + b_n x^{a_n} y^{b_n} = 0.$$

   [Hint: set $u = \ln x$ and $v = \ln y$.]

2. (Chapter 16 Review) Evaluate $\int_C x^3 y\, dx - x\, dy$, where C is the circle $x^2 + y^2 = 1$ with counterclockwise orientation.

3. (Chapter 16 Review) Show that the vector field $\mathbf{F}(x, y, z) = e^y \mathbf{i} + (xe^y + e^z)\mathbf{j} + ye^z \mathbf{k}$ is conservative. Then evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$, where C is the line segment from $(0, 2, 0)$ to $(4, 0, 3)$.

[2] Ed's main web page has links about the Putnam exam where you can find old exams and solutions.