Midterm 1 Review

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1 Parametric Equations and Polar Coordinates

1.1 Things to Know

- If $\frac{dx}{dt} \neq 0$, $\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$ and $\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}}$. Higher derivatives are computed in a similar manner.

- For a parametric curve $x = f(t)$, $y = g(t)$, $a \leq t \leq b$, the area under the curve is given by $\int_a^b g(t)f'(t)\, dt$. One thing to note about this formula is that it gives a number that depends on *how many times* your curve wraps around the area you're trying to find, as well as *which way* (that is, clockwise or counterclockwise).

- The arclength of a parametric curve where $a \leq t \leq b$ is given by

$$\int_a^b \sqrt{\left(\frac{dy}{dt}\right)^2 + \left(\frac{dx}{dt}\right)^2}\, dt.$$

  If you're trying to find the length of a closed curve, make sure you aren't wrapping around multiple times.

- The surface area obtained by rotating a parametric curve with $a \leq t \leq b$ is given by

$$\int_a^b 2\pi y\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt.$$

  Again, make sure you aren't wrapping around multiple times in the case of a closed curve. Here, there's also the danger that parts of your curve are symmetric about the x-axis. That can also cause this formula to give you numbers that aren't what you're looking for.

- To convert between polar to cartesian, $x = r\cos\theta$ and $y = \sin\theta$. Use these formulas most of the time.

- To convert from cartesian to polar, $r^2 = x^2 + y^2$ and $\tan\theta = \frac{y}{x}$. Use these formulas if you are given x and y in terms of a third variable such as t and you want to obtain r and θ in terms of t.

- Polar area, for a curve where $a \leq \theta \leq b$, is given by $\int_a^b \frac{1}{2}r^2\, d\theta$.

- Polar arclength, for a curve where $a \leq \theta \leq b$, is given by

$$\int_a^b \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta.$$

1.2 Problems

1. (Practice Midterm 1) Describe the planar curve given parametrically by $x = \cosh t$ and $y = \sinh t$ using a relation between x and y.
2. (Practice Midterm 1) Sketch the curve $r = 2 + 3\cos\theta$ and find the area enclosed by its inner loop.
3. (Practice Midterm 2) Sketch the curve $r = \cos^2\theta$ and find the area enclosed by it.
4. Sketch the curve $x = \tan t$, $y = \cot t$ by eliminating t.
5. Sketch the curve $x = 1 + e^{2t}$, $y = e^t$ by eliminating t.
6. Sketch the curve $r = 1 - 3\sin\theta$ and find the area enclosed by its inner loop.
7. Find the points of intersection of the curves $r = \cot\theta$ and $r = 2\cos\theta$.
8. Sketch the curve $r = \sin 4\theta$ and find the area it encloses.
9. Sketch the curve $r^2 = \sin 2\theta$ and find the area it encloses.
10. Sketch the curve $r^2\theta = 1$.

2 Vectors, Lines, and Planes

2.1 Things to Know

- $\mathbf{a} \cdot \mathbf{a} = |\mathbf{a}|^2$.
- $\mathbf{a} \cdot (\mathbf{b} + \mathbf{c}) = \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{c}$.
- $\mathbf{0} \cdot \mathbf{a} = 0$.
- $\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a}$.
- $(c\mathbf{a}) \cdot \mathbf{b} = c(\mathbf{a} \cdot \mathbf{b}) = \mathbf{a} \cdot (c\mathbf{b})$.
- If $\mathbf{a}$ and $\mathbf{b}$ are both nonzero and θ is the angle between them, then $\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}|\cos\theta$. This fact is used to derive formulas for the scalar and vector projections, which you should also know.
- $\mathbf{a}$ and $\mathbf{b}$ are perpendicular (or *orthogonal*) if and only if $\mathbf{a} \cdot \mathbf{b} = 0$.
- $\mathbf{a} \times \mathbf{b}$ is orthogonal to both $\mathbf{a}$ and $\mathbf{b}$, and its direction is determined by the right-hand rule.
- If $\mathbf{a}$ and $\mathbf{b}$ are both nonzero and θ is the angle between them, then $|\mathbf{a} \times \mathbf{b}| = |\mathbf{a}||\mathbf{b}|\sin\theta$.
- Two nonzero vectors $\mathbf{a}$ and $\mathbf{b}$ are parallel if and only if $\mathbf{a} \times \mathbf{b} = \mathbf{0}$.
- The area of a parallelogram is the magnitude of the cross product of two sides.
- $\mathbf{a} \times \mathbf{b} = -\mathbf{b} \times \mathbf{a}$.
- $(c\mathbf{a}) \times \mathbf{b} = c(\mathbf{a} \times \mathbf{b}) = \mathbf{a} \times (c\mathbf{b})$.
- $\mathbf{a} \times (\mathbf{b} + \mathbf{c}) = \mathbf{a} \times \mathbf{b} + \mathbf{a} \times \mathbf{c}$.
- $(\mathbf{a} + \mathbf{b}) \times \mathbf{c} = \mathbf{a} \times \mathbf{c} + \mathbf{b} \times \mathbf{c}$.
- $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}$.

- The scalar triple product is $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})$. It can be used to determine whether three vectors lie in a plane, or find the volume of a parallelepiped.
- $\mathbf{r} = \mathbf{r}_0 + t\mathbf{u}$ is the vector equation for a line passing through $\mathbf{r}_0$ and parallel to $\mathbf{u}$.
- The plane $ax + by + cz = d$ has normal vector $\langle a, b, c \rangle$.
- $\mathbf{n} \cdot (\mathbf{r} - \mathbf{r}_0) = 0$ is the equation for a plane passing through $\mathbf{r}_0$ with normal vector $\mathbf{n}$.
- Ellipsoid: $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$.
- Elliptic paraboloid: $\frac{z}{c} = \frac{x^2}{a^2} + \frac{y^2}{b^2}$.
- Hyperbolic paraboloid: $\frac{z}{c} = \frac{x^2}{a^2} - \frac{y^2}{b^2}$.
- Cone: $\frac{z^2}{c^2} = \frac{x^2}{a^2} + \frac{y^2}{b^2}$.
- Hyperboloid of one sheet: $\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$.
- Hyperboloid of two sheets: $\frac{x^2}{a^2} - \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$.
- Linear transformations can be applied to $\langle x, y, z \rangle$ for all of the above shapes. The basic type of shape will be preserved, but it will be rotated, stretched, whatever. Pay attention to constant terms; the last three types of shapes are distinguished *only* by the constant term you end up getting on the right-hand side of the equation.
- To convert between cylindrical and rectangular coordinates, use $x = r \cos\theta$, $y = r \sin\theta$, and $z = z$. Cylindrical coordinates is just polar coordinates in the xy-plane, and then a z-coordinate.
- To convert between spherical and rectangular coordinates, use $x = \rho \cos\theta \sin\phi$, $y = \rho \sin\theta \sin\phi$, and $z = \rho \cos\phi$. ρ is a greek "r," and stands for the radius of the point from the origin. θ is the angle with the x-axis, just like polar and cylindrical coordinates. ϕ is the angle with the positive z-axis.

2.2 Problems

1. (Practice Midterm 1) Suppose that $|\mathbf{a}| = 2$, $|\mathbf{b}| = 3$, and $\mathbf{a} \cdot \mathbf{b} = 3\sqrt{2}$. What is the area of the parallelogram spanned by $\mathbf{a}$ and $\mathbf{b}$?
2. (Practice Midterm 2) Suppose that $|\mathbf{a}| = 4$, $|\mathbf{b}| = 3$, and $\mathbf{a} \cdot \mathbf{b} = 6$. What is the length of $\mathbf{a} \times \mathbf{b}$?
3. (Practice Midterm 1) State the relations between the rectangular coordinates (x, y, z) and the cylindrical coordinates (r, θ, z) (accompany this by a figure). Sketch the solid described by $0 \le \theta \le \pi/2$, $r \le z \le 2$.
4. (Practice Midterm 2) State the relations between the rectangular coordinates (x, y, z) and the spherical coordinates (ρ, θ, ϕ) (accompany this by a figure). Sketch the solid described by $\rho \le 1$ and $\phi \ge \pi/3$.
5. (Practice Midterm 2) Show that the quadric

$$\frac{x^2}{8} + y^2 - \frac{z^2}{9} = 1$$

contains the curve

$$x = 2\cosh t, y = \frac{1}{\sqrt{2}}\cosh t, z = 3\sinh t,$$

and sketch the quadric (do not sketch the curve!). What is the name of this quadric?

6. For vectors **a**, **b**, and **c**, prove that

$$(\mathbf{a} \times \mathbf{b}) \cdot [(\mathbf{b} \times \mathbf{c}) \times (\mathbf{c} \times \mathbf{a})] = [\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})]^2 .$$

7. Find the acute angle between two diagonals of a cube.

8. Find the equation for the plane containing the points $(3, -1, 1)$, $(4, 0, 2)$, and $(6, 3, 1)$.

9. Find the distance between the planes $3x + y - 4z = 2$ and $3x + y - 4z = 24$.

10. Identify and sketch $-4x^2 + y^2 - 4z^2 = 4$.

11. Identify and sketch $4x^2 + 4y^2 - 8y + z^2 = 0$.

12. Identify the surface $\rho = 3 \sec \phi$.

3 Vector Functions

3.1 Things to Know

- The vector equation $\mathbf{r}(t) = \langle f(t), g(t), h(t) \rangle$ is equivalent to the parametric equations $x = f(t)$, $y = g(t)$, and $z = h(t)$.
- To take the limit of a vector function, take the limit of each component. The result is a vector. To differentiate a vector function, differentiate each component. The result is a vector. To integrate a vector function, integrate each component. The result is a vector.

3.2 Problems

1. (Practice Midterm 1) Consider the motion given by a position vector $\mathbf{r}(t) = \langle t^2 + 40t, 3t^2 + 1 \rangle$. At what time t is the velocity perpendicular to acceleration?

2. (a) Sketch the curve with vector function

$$\mathbf{r}(t) = t\mathbf{i} + (\cos \pi t)\mathbf{j} + (\sin \pi t)\mathbf{k}$$

   for $t \geq 0$.

   (b) Find $\mathbf{r}'(t)$ and $\mathbf{r}''(t)$. When are they perpendicular?

3. If $\mathbf{r}(t) = t^2\mathbf{i} + (t \cos \pi t)\mathbf{j} + (\sin \pi t)\mathbf{k}$, evaluate $\int_0^1 \mathbf{r}(t)\, dt$.

4 Partial Derivatives

4.1 Things to Know

- Definition of the limit of a multivariable function.
- Definitions of partial derivatives.
- Clairaut's Theorem: Suppose f is defined on a disk D that contains the point (a, b). If the functions f_{xy} and f_{yx} are both continuous on D, then $f_{xy}(a, b) = f_{yx}(a, b)$.
- Formula for tangent plane.
- Definition of differentiability of a multivariable function.
- If the partial derivatives f_x and f_y exist near (a, b) and are continuous at (a, b), then f is differentiable at (a, b).

4.2 Problems

1. (Practice Midterm 2) Show that the function $u(x, y) = y^{-\frac{1}{2}}e^{-x^2/(4y)}$ solves the following partial differential equation:

$$u_y = u_{xx}.$$

2. Evaluate each limit or show that it does not exist:

   (a) $\lim_{(x,y)\to(1,1)} \frac{2xy}{x^2+2y^2}$

   (b) $\lim_{(x,y)\to(0,0)} \frac{2xy}{x^2+2y^2}$

3. Find all second partial derivatives of $v(r, s, t) = r\cos(s + 2t)$.