Final Review

Edward Carter

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1 Parametric Equations and Polar Coordinates

1.1 Things to Know

- If $\frac{dx}{dt} \neq 0$, $\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$ and $\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}}$. Higher derivatives are computed in a similar manner.
- For a parametric curve $x = f(t)$, $y = g(t)$, $a \leq t \leq b$, the area under the curve is given by $\int_a^b g(t)f'(t)\,dt$. One thing to note about this formula is that it gives a number that depends on *how many times* your curve wraps around the area you're trying to find, as well as *which way* (that is, clockwise or counterclockwise).
- The arclength of a parametric curve where $a \leq t \leq b$ is given by

$$\int_a^b \sqrt{\left(\frac{dy}{dt}\right)^2 + \left(\frac{dx}{dt}\right)^2}\,dt.$$

  If you're trying to find the length of a closed curve, make sure you aren't wrapping around multiple times.
- The surface area obtained by rotating a parametric curve with $a \leq t \leq b$ is given by

$$\int_a^b 2\pi y\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt.$$

  Again, make sure you aren't wrapping around multiple times in the case of a closed curve. Here, there's also the danger that parts of your curve are symmetric about the x-axis. That can also cause this formula to give you numbers that aren't what you're looking for.
- To convert between polar to cartesian, $x = r\cos\theta$ and $y = \sin\theta$. Use these formulas most of the time.
- To convert from cartesian to polar, $r^2 = x^2 + y^2$ and $\tan\theta = \frac{y}{x}$. Use these formulas if you are given x and y in terms of a third variable such as t and you want to obtain r and θ in terms of t.
- Polar area, for a curve where $a \leq \theta \leq b$, is given by $\int_a^b \frac{1}{2}r^2\,d\theta$.
- Polar arclength, for a curve where $a \leq \theta \leq b$, is given by

$$\int_a^b \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\,d\theta.$$

1.2 Problems

1. (Practice Midterm 1 for Midterm 1) Describe the planar curve given parametrically by $x = \cosh t$ and $y = \sinh t$ using a relation between x and y.
2. (Practice Midterm 1 for Midterm 1) Sketch the curve $r = 2 + 3\cos\theta$ and find the area enclosed by its inner loop.
3. (Practice Midterm 2 for Midterm 1) Sketch the curve $r = \cos^2\theta$ and find the area enclosed by it.
4. (Chapter 10 Review) Sketch the curve $x = \tan t$, $y = \cot t$ by eliminating t.
5. (Chapter 10 Review) Sketch the curve $x = 1 + e^{2t}$, $y = e^t$ by eliminating t.
6. (Chapter 10 Review) Sketch the curve $r = 1 - 3\sin\theta$ and find the area enclosed by its inner loop.
7. (Chapter 10 Review) Find the points of intersection of the curves $r = \cot\theta$ and $r = 2\cos\theta$.
8. (Chapter 10 Review) Sketch the curve $r = \sin 4\theta$ and find the area it encloses.
9. Sketch the curve $r^2 = \sin 2\theta$ and find the area it encloses.
10. Sketch the curve $r^2\theta = 1$.

2 Vectors, Lines, and Planes

2.1 Things to Know

- $\mathbf{a} \cdot \mathbf{a} = |\mathbf{a}|^2$.
- $\mathbf{a} \cdot (\mathbf{b} + \mathbf{c}) = \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{c}$.
- $\mathbf{0} \cdot \mathbf{a} = 0$.
- $\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a}$.
- $(c\mathbf{a}) \cdot \mathbf{b} = c(\mathbf{a} \cdot \mathbf{b}) = \mathbf{a} \cdot (c\mathbf{b})$.
- If $\mathbf{a}$ and $\mathbf{b}$ are both nonzero and θ is the angle between them, then $\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}|\cos\theta$. This fact is used to derive formulas for the scalar and vector projections, which you should also know.
- $\mathbf{a}$ and $\mathbf{b}$ are perpendicular (or *orthogonal*) if and only if $\mathbf{a} \cdot \mathbf{b} = 0$.
- $\mathbf{a} \times \mathbf{b}$ is orthogonal to both $\mathbf{a}$ and $\mathbf{b}$, and its direction is determined by the right-hand rule.
- If $\mathbf{a}$ and $\mathbf{b}$ are both nonzero and θ is the angle between them, then $|\mathbf{a} \times \mathbf{b}| = |\mathbf{a}||\mathbf{b}|\sin\theta$.
- Two nonzero vectors $\mathbf{a}$ and $\mathbf{b}$ are parallel if and only if $\mathbf{a} \times \mathbf{b} = \mathbf{0}$.
- The area of a parallelogram is the magnitude of the cross product of two sides.
- $\mathbf{a} \times \mathbf{b} = -\mathbf{b} \times \mathbf{a}$.
- $(c\mathbf{a}) \times \mathbf{b} = c(\mathbf{a} \times \mathbf{b}) = \mathbf{a} \times (c\mathbf{b})$.
- $\mathbf{a} \times (\mathbf{b} + \mathbf{c}) = \mathbf{a} \times \mathbf{b} + \mathbf{a} \times \mathbf{c}$.
- $(\mathbf{a} + \mathbf{b}) \times \mathbf{c} = \mathbf{a} \times \mathbf{c} + \mathbf{b} \times \mathbf{c}$.
- $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}$.

- The scalar triple product is $\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})$. It can be used to determine whether three vectors lie in a plane, or find the volume of a parallelepiped.
- $\mathbf{r}=\mathbf{r}_0+t\mathbf{u}$ is the vector equation for a line passing through $\mathbf{r}_0$ and parallel to $\mathbf{u}$.
- The plane $ax+by+cz=d$ has normal vector $\langle a,b,c\rangle$.
- $\mathbf{n}\cdot(\mathbf{r}-\mathbf{r}_0)=0$ is the equation for a plane passing through $\mathbf{r}_0$ with normal vector $\mathbf{n}$.
- Ellipsoid: $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$.
- Elliptic paraboloid: $\frac{z}{c}=\frac{x^2}{a^2}+\frac{y^2}{b^2}$.
- Hyperbolic paraboloid: $\frac{z}{c}=\frac{x^2}{a^2}-\frac{y^2}{b^2}$.
- Cone: $\frac{z^2}{c^2}=\frac{x^2}{a^2}+\frac{y^2}{b^2}$.
- Hyperboloid of one sheet: $\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$.
- Hyperboloid of two sheets: $\frac{x^2}{a^2}-\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$.
- Linear transformations can be applied to $\langle x,y,z\rangle$ for all of the above shapes. The basic type of shape will be preserved, but it will be rotated, stretched, whatever. Pay attention to constant terms; the last three types of shapes are distinguished *only* by the constant term you end up getting on the right-hand side of the equation.
- To convert between cylindrical and rectangular coordinates, use $x=r\cos\theta$, $y=r\sin\theta$, and $z=z$. Cylindrical coordinates is just polar coordinates in the xy-plane, and then a z-coordinate.
- To convert between spherical and rectangular coordinates, use $x=\rho\cos\theta\sin\phi$, $y=\rho\sin\theta\sin\phi$, and $z=\rho\cos\phi$. ρ is a greek "r," and stands for the radius of the point from the origin. θ is the angle with the x-axis, just like polar and cylindrical coordinates. ϕ is the angle with the positive z-axis.

2.2 Problems

1. (Practice Midterm 1 for Midterm 1) Suppose that $|\mathbf{a}|=2$, $|\mathbf{b}|=3$, and $\mathbf{a}\cdot\mathbf{b}=3\sqrt{2}$. What is the area of the parallelogram spanned by $\mathbf{a}$ and $\mathbf{b}$?
2. (Practice Midterm 2 for Midterm 1) Suppose that $|\mathbf{a}|=4$, $|\mathbf{b}|=3$, and $\mathbf{a}\cdot\mathbf{b}=6$. What is the length of $\mathbf{a}\times\mathbf{b}$?
3. (Practice Midterm 1 for Midterm 1) State the relations between the rectangular coordinates (x,y,z) and the cylindrical coordinates (r,θ,z) (accompany this by a figure). Sketch the solid described by $0\le\theta\le\pi/2$, $r\le z\le 2$.
4. (Practice Midterm 2 for Midterm 1) State the relations between the rectangular coordinates (x,y,z) and the spherical coordinates (ρ,θ,ϕ) (accompany this by a figure). Sketch the solid described by $\rho\le 1$ and $\phi\ge\pi/3$.
5. (Practice Midterm 2 for Midterm 1) Show that the quadric
$$\frac{x^2}{8}+y^2-\frac{z^2}{9}=1$$
contains the curve
$$x=2\cosh t, y=\frac{1}{\sqrt{2}}\cosh t, z=3\sinh t,$$
and sketch the quadric (do not sketch the curve!). What is the name of this quadric?

6. (Chatper 12 Review) For vectors $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$, prove that

$$(\mathbf{a} \times \mathbf{b}) \cdot [(\mathbf{b} \times \mathbf{c}) \times (\mathbf{c} \times \mathbf{a})] = [\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})]^2 .$$

7. (Chapter 12 Review) Find the acute angle between two diagonals of a cube.
8. (Chapter 12 Review) Find the equation for the plane containing the points $(3, -1, 1)$, $(4, 0, 2)$, and $(6, 3, 1)$.
9. (Chapter 12 Review) Find the distance between the planes $3x + y - 4z = 2$ and $3x + y - 4z = 24$.
10. (Chapter 12 Review) Identify and sketch $-4x^2 + y^2 - 4z^2 = 4$.
11. (Chapter 12 Review) Identify and sketch $4x^2 + 4y^2 - 8y + z^2 = 0$.
12. (Chapter 12 Review) Identify the surface $\rho = 3 \sec \phi$.

3 Vector Functions

3.1 Things to Know

- The vector equation $\mathbf{r}(t) = \langle f(t), g(t), h(t) \rangle$ is equivalent to the parametric equations $x = f(t)$, $y = g(t)$, and $z = h(t)$.
- To take the limit of a vector function, take the limit of each component. The result is a vector. To differentiate a vector function, differentiate each component. The result is a vector. To integrate a vector function, integrate each component. The result is a vector.

3.2 Problems

1. (Practice Midterm 1 for Midterm 1) Consider the motion given by a position vector $\mathbf{r}(t) = \langle t^2 + 40t, 3t^2 + 1 \rangle$. At what time t is the velocity perpendicular to acceleration?
2. (Chapter 13 Review)

   (a) Sketch the curve with vector function

   $$\mathbf{r}(t) = t\mathbf{i} + (\cos \pi t)\mathbf{j} + (\sin \pi t)\mathbf{k}$$

   for $t \geq 0$.

   (b) Find $\mathbf{r}'(t)$ and $\mathbf{r}''(t)$. When are they perpendicular?
3. (Chapter 13 Review) If $\mathbf{r}(t) = t^2\mathbf{i} + (t \cos \pi t)\mathbf{j} + (\sin \pi t)\mathbf{k}$, evaluate $\int_0^1 \mathbf{r}(t)\, dt$.

4 Partial Derivatives

4.1 Things to Know

- Definition of the limit of a multivariable function.
- Definitions of partial derivatives.
- Clairaut's Theorem: Suppose f is defined on a disk D that contains the point (a, b). If the functions f_{xy} and f_{yx} are both continuous on D, then $f_{xy}(a, b) = f_{yx}(a, b)$.

- Formula for tangent plane.
- Definition of differentiability of a multivariable function.
- If the partial derivatives f_x and f_y exist near (a, b) and are continuous at (a, b), then f is differentiable at (a, b).
- If $z = f(x, y)$, $x = g(t)$, and $y = h(t)$, then

$$\frac{dz}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}.$$

- If $z = f(x, y)$, $x = g(s, t)$, and $y = h(s, t)$, then

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s}$$

  and

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t}.$$

- If $u = f(x_1, \ldots, x_n)$ and each x_i is a differentiable function of $t_1, \ldots, t_m$, then

$$\frac{\partial u}{\partial t_i} = \sum_{j=1}^{n} \frac{\partial u}{\partial x_j}\frac{\partial x_j}{\partial t_i}.$$

- Let f be a function of x and y. The gradient of f at (a, b), $\nabla f(a, b)$, is defined to be the vector $\langle f_x(a, b), f_y(a, b)\rangle$. The definition is analogous for functions with more than two variables. The gradient vector points in the direction of steepest ascent for f. The vector $\nabla f(a, b)$ is always perpendicular to the level curve passing through (a, b).
- The directional derivative of a function of two variables is the derivative of the one variable function you get by taking a slice of the two variable function along a certain direction. It can be computed by taking the dot product of a unit vector for the direction you're looking at and the gradient of the function at that point.
- A function $f(x, y, z)$ of three variables has level surfaces rather than level curves. At any point (a, b, c), the normal vector for the plane tangent to the level surface of f passing through (a, b, c) is given by $\nabla f(a, b, c)$.
- At any local minimum or maximum of any function, the gradient either does not exist or is equal to the zero vector. Points where the gradient vector does not exist or is equal to the zero vector are called critical points, like in 1A.
- Let $f(x, y)$ have continuous partial derivatives on a disk centered at (a, b), and suppose $f_x(a, b) = f_y(a, b) = 0$. Let $D = f_{xx}(a, b)f_{yy}(a, b) - [f_{xy}(a, b)]^2$. Then if $D > 0$ and $f_{xx}(a, b) > 0$, then (a, b) is a local maximum for f. If $D > 0$ and $f_{xx}(a, b) < 0$, then (a, b) is a local minimum. If $D < 0$, then (a, b) is neither a local maximum nor a local minimum.
- Recall from 1A that if you have a continuous function on a closed and bounded set, it must achieve maximum and minimum values. You only need to check the boundary points and the critical points. This applies to functions in any number of variables as well, except that the boundaries and critical points can now be more complicated.

- To maximize $f(x,y,z)$ subject to the constraint $g(x,y,z)=k$, find all values of x, y, z, and λ such that $\nabla f(x,y,z)=\lambda\nabla g(x,y,z)$ and $g(x,y,z)=k$. Then you only need to try these points (ignoring λ).
- To maximize $f(x,y,z)$ subject to the constraints $g(x,y,z)=k$ and $h(x,y,z)=s$, find all values of x, y, z, λ, and μ such that $\nabla f(x,y,z)=\lambda\nabla g(x,y,z)+\mu\nabla h(x,y,z)$, $g(x,y,z)=k$ and $h(x,y,z)=s$. Then you only need to try these points (ignoring λ and μ).

4.2 Problems

1. (Practice Midterm 2 for Midterm 1) Show that the function $u(x,y)=y^{-\frac{1}{2}}e^{-x^2/(4y)}$ solves the following partial differential equation:

$$u_y = u_{xx}.$$

2. (Chapter 14 Review) Evaluate each limit or show that it does not exist:
   (a) $\lim_{(x,y)\to(1,1)} \frac{2xy}{x^2+2y^2}$
   (b) $\lim_{(x,y)\to(0,0)} \frac{2xy}{x^2+2y^2}$
3. (Chapter 14 Review) Find all second partial derivatives of $v(r,s,t)=r\cos(s+2t)$.
4. (Spring 1993 Prelim[1]) Prove that $\frac{x^2+y^2}{4}\le e^{x+y-2}$ for $x\ge 0$, $y\ge 0$.
5. (Spring 2003 Prelim) Define $f:\mathbb{R}^2\to\mathbb{R}$ by $f(x,0)=0$ and

$$f(x,y)=\left[1-\cos\left(\frac{x^2}{y}\right)\right]\sqrt{x^2+y^2}$$

for $y\neq 0$.
   (a) Show that f is continuous at $(0,0)$.
   (b) Calculate all the directional derivatives of f at $(0,0)$.
   (c) Show that f is not differentiable at $(0,0)$.
6. (a) Find the minimal value of the areas of triangles circumscribing the unit circle in $\mathbb{R}^2$.
   (b) (Fall 1998 Prelim) Find the minimal value of the areas of hexagons circumscribing the unit circle in $\mathbb{R}^2$. (This problem requires calculus of six variables. The previous part only requires three variables.)
7. (Chapter 14 Review) Find the absolute minimum and maximum values of $f(x,y)=e^{-x^2-y^2}(x^2+2y^2)$ on the set $\{(x,y)\in\mathbb{R}^2 \mid x^2+y^2\le 4\}$.
8. (Practice Midterm 1 for Midterm 2) Find the partial derivative $\frac{\partial u}{\partial \xi}$ in terms of the partial derivatives of $u(x,y,z,w)$, where, as a function of (ξ,η,ζ),

$$u=u(\xi, e^{\xi+\eta}, (1+\xi^2+\eta^2+\zeta^2)^{-\frac{1}{2}}, \xi-\eta).$$

9. (Practice Midterm 1 for Midterm 2) Show that the triangle with a maximal area and with the given length of the perimeter, p, is equilateral. (Hint: Use Heron's formula.)

[1]A website for the prelim can be found on the math department's web page, by clicking on "Graduate" and then on "Prelim Exam." This site has old exams and some solutions. There is also a book called *Berkeley Problems in Mathematics*, which most GSIs have access to, that has solutions to all prelim problems appearing in this document.

10. (Practice Midterm 1 for Midterm 2) Find all the absolute maxima and minima of the function

$$(x^2 + 2xy + 3y^2 - 5)\sin^2 z,$$

on the closed (unbounded) set

$$\{(x, y, z) \in \mathbb{R}^3 \mid (x, y) \in D, z \in \mathbb{R}\},$$

where D is the closed triangular region with vertices at $(-1, 1)$, $(2, 1)$, and $(-1, -2)$.

11. (Practice Midterm 2 for Midterm 2) Consider the function

$$f(x, y) = x^2 + \frac{x^3}{3} - xy^2 + y^2.$$

Find the absolute minimum and maximum of f inside the unit disc $x^2 + y^2 \leq 1$.

12. (Chapter 14 Review) Find the points on the surface $xy^2z^3 = 2$ that are closest to the origin.

13. (Chapter 14 Review) A package in the shape of a rectangular box can be mailed by U.S. Parcel Post if the sum of its length and girth (the perimeter of a cross-section perpendicular to the length) is at most 108 in. Find the dimensions of the package with largest volume that can be mailed by Parcel Post.

14. Find the absolute maximum and maximum of the function

$$f(x, y, z) = e^{-z^2}|x| \sin y \cos(x + y)$$

on the set $\{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 \leq \pi^2\}$.

5 Multiple Integrals

5.1 Things to Know

- Definitions of double and triple integrals (they are analogous to the single variable case). Same with things like the midpoint rules.
- Fubini's Theorem. (The "iterated integrals exist" means that you get a finite value when you evaluate the iterated integrals with absolute value signs around the function being integrated.)
- How to integrate things over complicated regions.
- If $D = D_1 \cup D_2$ and D_1 and D_2 don't overlap except on their boundaries, then $\int\int_D f(x, y)\, dA = \int\int_{D_1} f(x, y)\, dA + \int\int_{D_2} f(x, y)\, dA$.
- The area of D is $\int\int_D \, dA$.
- If $m \leq f(x, y) \leq M$ for all $(x, y) \in D$, then $mA(D) \leq \int\int_D f(x, y)\, dA \leq MA(D)$.
- $dA = r\, dr\, d\theta$.
- To find the mass of an object, m, take the integral of the density function, ρ over the object. The x-coordinate of the center of mass is given by $\overline{x} = \frac{1}{m}\int\int_D x\rho\, dA$. Similarly for the y-coordinate.
- Moment of inertia stuff, expected value stuff.
- Most of the stuff above applies to triple integrals, too.

- The surface area of a surface is given by

$$A(S) = \int\int_D \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\, dA.$$

- $dV = \rho^2 \sin\phi \; d\rho \; d\phi \; d\theta = r \; dr \; d\theta \; dz.$
- The Jacobian of a transformation given by $x = g(u, v)$ and $y = h(u, v)$ is

$$\frac{\partial(x, y)}{\partial(u, v)} = \det \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{pmatrix}.$$

  It's similar for a three-dimensional transformation.

- Let T be a C^1 transformation mapping R to S. Under some other conditions (look them up),

$$\int\int_R f(x, y) \; dA = \int\int_S f(x(u, v), y(u, v)) \left| \frac{\partial(x, y)}{\partial(u, v)} \right| \; du \; dv.$$

  There is a similar formula for three variables.

5.2 Problems

1. (Spring 1978 Prelim) What is the volume enclosed by the ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1?$$

2. (Spring 1978 Prelim) Evaluate

$$\int\int_{\mathcal{A}} e^{-x^2-y^2} \; dx \; dy,$$

   where $\mathcal{A} = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq 1\}$.

3. (Spring 1998 Prelim) Given the fact that $\int_{-\infty}^{\infty} e^{-x^2} \; dx = \sqrt{\pi}$, evaluate the integral

$$I = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-(x^2+(y-x)^2+y^2)} \; dx \; dy.$$

4. (Practice Midterm 1 for Midterm 2) Evaluate $\int\int_D (1/x) \; dA$, where $D = \{(x, y) \mid 1 \leq y \leq e, y^2 \leq x \leq y^4\}$.
5. (Practice Midterm 1 for Midterm 2) Evaluate $\int_0^1 \int_{3y}^3 e^{x^2} \; dx \; dy$.
6. (Practice Midterm 1 for Midterm 2) Evaluate the Jacobian

$$\frac{\partial(x, y, z)}{\partial(u, v, w)}$$

   of the transformation

$$x = e^u + e^v,$$
$$y = e^u - e^v,$$
$$z = u + v.$$

7. (Practice Midterm 1 for Midterm 2) Find the surface area of the set

$$\{(x, y, z) \in \mathbb{R}^3 \mid z = x + y^2, 0 \leq x \leq ay, 0 \leq y \leq b\}$$

in terms of a and b.

8. (Practice Midterm 2 for Midterm 2) Evaluate $\int\int_D e^x \, dA$, where D is the triangle with vertices at $(0, 0)$, $(2, 4)$, and $(6, 0)$.

9. (Practice Midterm 2 for Midterm 2) Evaluate $\int_0^1 \int_{x^2}^1 x^3 \sin\left(y^3\right) \, dy \, dx$.

10. (Practice Midterm 2 for Midterm 2) Evaluate the Jacobian of the transformation

$$\begin{aligned} x &= u + v^2 + w \\ y &= v + w^4 \\ z &= u - w. \end{aligned}$$

11. (Practice Midterm 2 for Midterm 2) Consider the following solid in $\mathbb{R}^3$:

$$R = \{(x, y, z) \in \mathbb{R}^3 \mid 0 \leq x \leq 1, x \leq y \leq 1, 0 \leq z \leq x + y^2\}.$$

Find the volume of R and the area of the surface which forms its boundary.

12. (Chapter 15 Review) Use spherical coordinates to evaluate

$$\int_{-2}^{2} \int_{0}^{\sqrt{4-y^2}} \int_{-\sqrt{4-x^2-y^2}}^{\sqrt{4-x^2-y^2}} y^2 \sqrt{x^2 + y^2 + z^2} \, dz \, dx \, dy$$

13. (Chapter 15 Review) Give five other iterated integrals that are equal to

$$\int_0^2 \int_0^{y^3} \int_0^{y^2} f(x, y, z) \, dz \, dx \, dy$$

14. (Chapter 15 Review) Use the transformation $x = u^2$, $y = v^2$, $z = w^2$ to find the volume of the region bounded by the surface $\sqrt{x} + \sqrt{y} + \sqrt{z} = 1$ and the coordinate planes.

15. (Chapter 15 Review) Use the change of variables formula and an appropriate transformation to evaluate $\int\int_R xy \, dA$, where R is the square with vertices $(0, 0)$, $(1, 1)$, $(2, 0)$, and $(1, -1)$.

16. Consider the region bounded by $z = x^2 + y^2$ and $x + y + z = 1$. Write an integral over this region as all six possible equivalent iterated integrals.

17. (Folland[2] Section 2.6) For each of the following functions f, compute the integrals $\int_0^1 \int_0^1 f(x, y) \, dx \, dy$ and $\int_0^1 \int_0^1 f(x, y) \, dy \, dx$. How does this relate to Fubini's Theorem?

    (a) $f(x, y) = \frac{x^2 - y^2}{(x^2 + y^2)^2}$.

    (b) $f(x, y) = (1 - xy)^{-a}$, where $a > 0$.

    (c) $f(x, y) = (x - 1/2)^{-3}$ if $0 < y < |x - 1/2|$, $f(x, y) = 0$ otherwise. In other words, compute the iterated integrals of $f(x, y) = (x - 1/2)^{-3}$ over the region $\{(x, y) \in [0, 1] \times [0, 1] \mid 0 < y < |x - 1/2|\}$.

[2] *Real Analysis: Modern Techniques and Their Applications* by Gerald Folland.

6 Vector Calculus

6.1 Things to Know

- The terms "vector field" and "conservative vector field."
- Definitions of line integrals.
- Simply-connectedness and the theorems in 16.3.
- Green's Theorem!
- Definitions of curl, divergence, and Laplacian.
- If f is a function of three variables with continuous second-order partial derivatives, then $\text{curl}\,(\nabla f) = \mathbf{0}$.
- If $\mathbf{F}$ is a vector field defined on all of $\mathbb{R}^3$ (or an open, simply-connected subset) whose component functions have continuous partial derivatives and $\text{curl}\,\mathbf{F} = \mathbf{0}$, then $\mathbf{F}$ is a conservative vector field.
- If $\mathbf{F}$ is a vector field whose components have continuous second-order partial derivatives, then $\text{div}\ \text{curl}\,\mathbf{F} = 0$.
- Vector forms of Green's Theorem.
- Here are some identities which are exercises in section 16.5.
  - $\text{div}\,(\mathbf{F}+\mathbf{G}) = \text{div}\,\mathbf{F} + \text{div}\,\mathbf{G}$
  - $\text{curl}\,(\mathbf{F}+\mathbf{G}) = \text{curl}\,\mathbf{F} + \text{curl}\,\mathbf{G}$
  - $\text{div}\,(f\mathbf{F}) = f\ \text{div}\,\mathbf{F} + \mathbf{F}\cdot\nabla f$
  - $\text{curl}\,(f\mathbf{F}) = f\ \text{curl}\,F + (\nabla f)\times\mathbf{F}$
  - $\text{div}\,(\mathbf{F}\times\mathbf{G}) = \mathbf{G}\cdot\ \text{curl}\,\mathbf{F} - \mathbf{F}\cdot\ \text{curl}\,\mathbf{G}$
  - $\text{div}\,(\nabla f \times \nabla g) = 0$
  - $\text{curl}\,(\ \text{curl}\,\mathbf{F}) = \nabla(\ \text{div}\,\mathbf{F}) - \nabla^2\mathbf{F}$
- What a parametric surface is, and how to find its area.
- Surface integrals.
- Flux integrals.
- Stokes' Theorem!
- Divergence Theorem!

6.2 Problems

1. (1996 Putnam[3]—VERY TRICKY) Let $(a_1,b_1),(a_2,b_2),\ldots,(a_n,b_n)$ be the vertices of a convex polygon which contains the origin in its interior. Prove that there exist positive real numbers x and y such that both of the following equations hold:

$$a_1x^{a_1}y^{b_1} + a_2x^{a_2}y^{b_2} + \cdots + a_nx^{a_n}y^{b_n} = 0$$
$$b_1x^{a_1}y^{b_1} + b_2x^{a_2}y^{b_2} + \cdots + b_nx^{a_n}y^{b_n} = 0.$$

   [Hint: set $u = \ln x$ and $v = \ln y$.]

[3] Ed's main web page has links about the Putnam exam where you can find old exams and solutions.

2. (Chapter 16 Review) Evaluate $\int_C x^3 y\, dx - x\, dy$, where C is the circle $x^2+y^2=1$ with counterclockwise orientation.

3. (Chapter 16 Review) Show that the vector field $\mathbf{F}(x,y,z) = e^y \mathbf{i} + (xe^y + e^z)\mathbf{j} + ye^z\mathbf{k}$ is conservative. Then evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$, where C is the line segment from $(0,2,0)$ to $(4,0,3)$.

4. (Chapter 16 Review) Show that there is no vector field $\mathbf{G}$ such that $\operatorname{curl} \mathbf{G} = 2x\mathbf{i} + 3yz\mathbf{j} - xz^2\mathbf{k}$.

5. (Chapter 16 Review) Find a nice formula for $\operatorname{curl}(\mathbf{F} \times \mathbf{G})$. Also, give the conditions on $\mathbf{F}$ and $\mathbf{G}$ for this formula to hold.

6. (Chapter 16 Review) If f is a harmonic function, that is, $\nabla^2 f = 0$, show that the line integral $\int f_y\, dx - f_x\, dy$ is independent of path in any simple region D.

7. (Spring 1980 Prelim) Let $\mathcal{S} = \{(x,y,z) \in \mathbb{R}^3 \mid x^2+y^2+z^2 = 1\}$ denote the unit sphere in $\mathbb{R}^3$. Evaluate the surface integral over $\mathcal{S}$:

$$\int\int_{\mathcal{S}} (x^2 + y + z)\, dS.$$

8. (Spring 1981 Prelim) Let $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$ be the usual unit vectors in $\mathbf{R}^3$. Let $\mathbf{F}$ denote the vector field

$$(x^2 + y - 4)\mathbf{i} + 3xy\mathbf{j} + (2xz + z^2)\mathbf{k}.$$

   (a) Compute $\operatorname{curl} \mathbf{F}$.

   (b) Compute the integral of $\operatorname{curl} \mathbf{F}$ over the surface $x^2+y^2+z^2 = 16$, $z \geq 0$.

9. (Chapter 16 Review) Compute $\int\int_S \mathbf{F} \cdot d\mathbf{S}$, where $\mathbf{F}(x,y,z) = xz\mathbf{i} - 2y\mathbf{j} + 3x\mathbf{k}$ and S is the sphere $x^2+y^2+z^2 = 4$ with outward orientation.

10. (Chapter 16 Review) Compute $\int\int_S \mathbf{F} \cdot d\mathbf{S}$, where $\mathbf{F}(x,y,z) = x^2\mathbf{i} + xy\mathbf{j} + z\mathbf{k}$ and S is the part of the paraboloid $z = x^2+y^2$ below the plane $z=1$ with upward orientation.

11. Chapter 16 Review, problem 38.

12. Chapter 16 Review, problem 39.

13. (Chapter 16 Problems Plus) Find the positively oriented simple closed curve C for which the value of the line integral

$$\int_C (y^3 - y)\, dx - 2x^3\, dy$$

is a maximum.

14. Give a parametrization of a doughnut.

15. Give a parametrization of a hyperboloid of one sheet.

16. Give a parametrization of an ellipsoid.

17. Compute the flux of the vector field

$$\mathbf{F}(x,y,z) = \left\langle x\cos z - ye^{yz}, xy\cos(xyz) - y\cos z, \frac{x^2+y^2+z^2}{z} - xz\cos(xyz) \right\rangle$$

across the surface $x^2+y^2+z^2 = 4$, oriented outward. [Hint: find the curl of the vector field $\langle \sin(xyz), e^{yz}, xy\cos z \rangle$.]