1. Find the general solution to the following differential equation.

$$\mathbf{x}'(t) = \begin{pmatrix} 2 & 1 \\ 0 & 1 \end{pmatrix} \mathbf{x}(t) + \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Since the 2×2 matrix is invertible, 0 is not one of its eigenvalues, so we can let

$$\mathbf{x}_p(t) = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}.$$

Then $\mathbf{x}_p'(t) = \mathbf{0}$, so

$$\mathbf{x}_p(t) = -\begin{pmatrix} 2 & 1 \\ 0 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = -\frac{1}{2}\begin{pmatrix} 1 & -1 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1/2 \\ -1 \end{pmatrix}.$$

By inspection, we see that the characteristic polynomial of the 2×2 matrix is $(\lambda - 2)(\lambda - 1)$ and that the eigenspace for the eigenvalue 2 is spanned by $\begin{pmatrix} 1 & 0 \end{pmatrix}^T$. The eigenspace for the eigenvalue 1 is

$$NS \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} = \text{span}\left\{ \begin{pmatrix} -1 \\ 1 \end{pmatrix} \right\}.$$

Therefore the general solution is

$$\mathbf{x}(t) = \begin{pmatrix} 1/2 \\ -1 \end{pmatrix} + c_1 e^{2t} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + c_2 e^{t} \begin{pmatrix} -1 \\ 1 \end{pmatrix}.$$

2. Let $A = \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix}$. Compute e^{At}.

First, observe that $A^2 = 0$, which follows from Cayley-Hamilton. Then

$$e^{At} = \sum_{n=0}^{\infty} \frac{A^n t^n}{n!} = I + At = \begin{pmatrix} 1+t & -t \\ t & 1-t \end{pmatrix}.$$