1. Find an orthogonal matrix P and a diagonal matrix D such that $A = PDP^T$.

$$A = \begin{pmatrix} -2 & 1 & -2 \\ 1 & -2 & -2 \\ -2 & -2 & 1 \end{pmatrix}$$

$$\begin{aligned} \det(A - \lambda I) &= \begin{vmatrix} -2-\lambda & 1 & -2 \\ 1 & -2-\lambda & -2 \\ -2 & -2 & 1-\lambda \end{vmatrix} \\ &= -(\lambda+2)(\lambda^2+\lambda-6) - (-\lambda-3) - 2(-2\lambda-6) \\ &= (\lambda+3)(4-\lambda^2+1+4) \\ &= -(\lambda+3)^2(\lambda-3) \end{aligned}$$

$$NS(A-3I) = NS\begin{pmatrix} -5 & 1 & -2 \\ 1 & -5 & -2 \\ -2 & -2 & -2 \end{pmatrix} = NS\begin{pmatrix} 1 & -5 & -2 \\ 0 & -12 & -6 \\ 0 & -24 & -12 \end{pmatrix} = NS\begin{pmatrix} 1 & -1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 0 \end{pmatrix} = \left\{ \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix} \right\}$$

$$NS(A+3I) = NS\begin{pmatrix} 1 & 1 & -2 \\ 1 & 1 & -2 \\ -2 & -2 & 4 \end{pmatrix} = \left\{ \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\}$$

These three eigenvectors are already orthogonal, so we only need to normalize them.

$$P = \begin{pmatrix} 1/\sqrt{6} & 1/\sqrt{2} & 1/\sqrt{3} \\ 1/\sqrt{6} & -1/\sqrt{2} & 1/\sqrt{3} \\ -1/\sqrt{3} & 0 & 1/\sqrt{3} \end{pmatrix}$$

$$D = \begin{pmatrix} 3 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & -3 \end{pmatrix}$$

2. Solve the following initial value problem.

$$\begin{aligned} y'' - 4y' + 3y &= 0 \\ y(0) &= 1 \\ y'(0) &= 1 \end{aligned}$$

The general solution is $y(t) = c_1e^t + c_2e^{3t}$. The initial conditions tell us that $1 = c_1 + c_2$ and $1 = c_1 + 3c_2$. Therefore $c_1 = 0$ and $c_2 = 0$, so $y(t) = e^t$.