1. Find an orthogonal matrix P and a diagonal matrix D such that $A = PDP^T$.

$$A = \begin{pmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{pmatrix}$$

The characteristic polynomial of A is

$$\begin{aligned} \det(A - \lambda I) &= \begin{vmatrix} -1-\lambda & 2 & 2 \\ 2 & -1-\lambda & 2 \\ 2 & 2 & -1-\lambda \end{vmatrix} \\ &= -(\lambda+1)(\lambda^2 + 2\lambda - 3) + 2(2\lambda + 6) + 2(2\lambda + 6) \\ &= (\lambda + 3)(1 - \lambda^2 + 8) \\ &= (\lambda+3)^2(\lambda - 3). \end{aligned}$$

The eigenspace for $\lambda = 3$ is

$$NS(A - 3I) = NS\begin{pmatrix} -4 & 2 & 2 \\ 2 & -4 & 2 \\ 2 & 2 & -4 \end{pmatrix} = \text{span}\left\{\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}\right\}.$$

The eigenspace for $\lambda = -3$ is

$$NS(A + 3I) = NS\begin{pmatrix} 2 & 2 & 2 \\ 2 & 2 & 2 \\ 2 & 2 & 2 \end{pmatrix} = \text{span}\left\{\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}\right\}.$$

Therefore we can take

$$P = \begin{pmatrix} 1/\sqrt{3} & 1/\sqrt{2} & 1/\sqrt{6} \\ 1/\sqrt{3} & -1/\sqrt{2} & 1/\sqrt{6} \\ 1/\sqrt{3} & 0 & -2/\sqrt{6} \end{pmatrix}$$

and

$$D = \begin{pmatrix} 3 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & -3 \end{pmatrix}.$$

2. Solve the following initial value problem.

$$\begin{aligned} y'' - 2y' + y &= 0 \\ y(0) &= 0 \\ y'(0) &= 1 \end{aligned}$$

The general solution is

$$y = c_1 e^t + c_2 t e^t$$

and

$$y' = c_1 e^t + c_2(t+1)e^t.$$

Therefore $c_1 = 0$ and $c_1 + c_2 = 1$. Thus the solution to the initial value problem is $y = te^t$.