1. Find the Fourier series for the function f with period 2π defined on the interval $[-\pi, \pi]$ by:

$$f(x) = \begin{cases} \pi, & |x| \leq \frac{\pi}{3} \\ -\pi, & \frac{\pi}{3} < |x| \leq \pi \end{cases}$$

Since f is an even function, $b_n = 0$ for all n, and

$$\begin{aligned} a_n &= \frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi x}{L}\right)\,dx \\ &= \frac{2}{\pi}\left[\int_0^{\pi/3} \pi\cos(nx)\,dx + \int_{\pi/3}^{\pi}(-\pi)\cos(nx)\,dx\right] \\ &= 2\left[\int_0^{\pi/3}\cos(nx)\,dx - \int_{\pi/3}^{\pi}\cos(nx)\,dx\right]. \end{aligned}$$

Then if $n > 0$,

$$a_n = 2\left[\left(\frac{\sin(nx)}{n}\right)\Big|_{x=0}^{\pi/3} - \left(\frac{\sin(nx)}{n}\right)\Big|_{x=\pi/3}^{\pi}\right] = \frac{4}{n}\sin\left(\frac{n\pi}{3}\right),$$

and

$$a_0 = 2\left[\frac{\pi}{3} - \frac{2\pi}{3}\right] = -\frac{2\pi}{3}.$$

Therefore the Fourier series for f is

$$-\frac{\pi}{3} + 4\sum_{n=1}^{\infty}\frac{1}{n}\sin\left(\frac{n\pi}{3}\right)\cos(nx).$$

2. Describe the convergence of the Fourier series in the previous problem: for which values of x does the series converge, and for those x, to what value does the series converge?

Since f and f' are both piecewise continuous, the Fourier series converges everywhere and is also periodic with period 2π. On the interval $[-\pi, \pi]$, it converges to the following function:

$$g(x) = \begin{cases} \pi, & |x| < \frac{\pi}{3} \\ 0, & |x| = \frac{\pi}{3} \\ -\pi, & \frac{\pi}{3} < |x| \leq \pi \end{cases}$$