

Theorem 1. *Let L and M be subspaces of a vector space V . If $L \cup M$ is also a subspace of V , then either $L \subseteq M$ or $M \subseteq L$.*

Proof. Suppose that $L \cup M$ is a subspace of V , $L \not\subseteq M$, and $M \not\subseteq L$. Then there exist $\mathbf{l} \in L$ and $\mathbf{m} \in M$ such that $\mathbf{l} \notin M$ and $\mathbf{m} \notin L$. However, both \mathbf{l} and \mathbf{m} are contained in $L \cup M$. Since $L \cup M$ is a subspace of V , $L \cup M$ is closed under addition, so $\mathbf{l} + \mathbf{m} \in L \cup M$. Let $\mathbf{v} = \mathbf{l} + \mathbf{m}$. Then either $\mathbf{v} \in L$ or $\mathbf{v} \in M$. If $\mathbf{v} \in L$, then since L is closed under addition and multiplication by the scalar -1 , $\mathbf{m} = \mathbf{v} - \mathbf{l} \in L$, which is a contradiction. On the other hand, if $\mathbf{v} \in M$, M is also closed under addition and multiplication by the scalar -1 , so $\mathbf{l} = \mathbf{v} - \mathbf{m} \in M$, which is also a contradiction. Therefore, if $L \cup M$ is a subspace of V , then either $L \subseteq M$ or $M \subseteq L$. \square

In particular, if L and M are distinct lines through the origin in \mathbb{R}^2 , then $L \cup M$ is not a vector space.

Theorem 2. *Let L and M be subspaces of a vector space V . Then $L \cap M$ is also a subspace of V .*

Proof. We will use the subspace test. Let $\mathbf{u}, \mathbf{v} \in L \cap M$, and let r be any scalar. Then \mathbf{u} and \mathbf{v} are each contained in both L and M . Since L is closed under addition, $\mathbf{u} + \mathbf{v} \in L$. Since M is closed under addition, $\mathbf{u} + \mathbf{v} \in M$. Therefore $\mathbf{u} + \mathbf{v} \in L \cap M$, so $L \cap M$ is closed under addition.

Since L is closed under scalar multiplication, $r\mathbf{u} \in L$. Since M is closed under scalar multiplication, $r\mathbf{u} \in M$. Therefore $r\mathbf{u} \in L \cap M$, so $L \cap M$ is closed under scalar multiplication.

Therefore $L \cap M$ is a subspace of V . \square

One can repeatedly apply Theorem 2 to find that any finite intersection of subspaces of a fixed vector space V is a subspace of V . It is also true that infinite intersections of subspaces of V are subspaces of V .