

We are given the differential equation

$$t^2(t+3)y''' - 3t(t+2)y'' + 6(1+t)y' - 6y = 0 \quad (1)$$

and two solutions, $y_1(t) = t^2$ and $y_2(t) = t^3$. The method of reduction of order from problem 26 starts with a differential equation of the form

$$y''' + p_1(t)y'' + p_2(t)y' + p_3(t)y = 0.$$

Given a solution y_1 , we apply the substitution $y = y_1(t)v(t)$ to obtain the second-order differential equation

$$y_1v''' + (3y_1' + p_1y_1)v'' + (3y_1'' + 2p_1y_1' + p_2y_1)v' = 0$$

in the variable v' . Applying this to (1) with the solution $y_1(t) = t^2$, we obtain

$$t^2v''' + \left(6t - \frac{3t(t+2)}{t+3}\right)v'' + \left(6 - \frac{12(t+2)}{t+3} + \frac{6(1+t)}{t+3}\right)v' = 0. \quad (2)$$

Since

$$6 - \frac{12(t+2)}{t+3} + \frac{6(1+t)}{t+3} = \frac{6t+18-12t-24+6t+6}{t+3} = 0$$

and

$$6t - \frac{3t(t+2)}{t+3} = \frac{6t^2+18t-3t^2-6t}{t+3} = \frac{3t(t+4)}{t+3},$$

the equation (2) reduces to

$$t^2u' + \left(\frac{3t(t+4)}{t+3}\right)u = 0,$$

where $u = v''$. Then

$$\frac{u'}{u} = -\frac{3(t+4)}{t(t+3)} = \frac{1}{t+3} - \frac{4}{t}$$

so that

$$\ln u = \ln(t+3) - 4 \ln t + C$$

and

$$u = \frac{A}{t^3} + \frac{3A}{t^4}.$$

Therefore

$$v = \frac{A}{2t} + \frac{A}{2t^2} + Bt + D$$

which gives

$$y = \frac{A}{2}(t+1) + Bt^3 + Dt^2,$$

the general solution to (1).