Theorem 1. *Let $T : V \to W$ be a linear transformation. Then the set*

$$T(V) = \{T(\mathbf{v}) \mid \mathbf{v} \in V\},$$

which we call the image of T, is a subspace of W.

Proof. Let $\mathbf{w}, \mathbf{x} \in T(V)$ and $\alpha \in \mathbb{R}$. Then there exist $\mathbf{u}, \mathbf{v} \in V$ such that $T(\mathbf{u}) = \mathbf{w}$ and $T(\mathbf{v}) = \mathbf{x}$. Then, since T is linear,

$$T(\mathbf{u} + \mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v}) = \mathbf{w} + \mathbf{x}$$

and

$$T(\alpha\mathbf{u}) = \alpha T(\mathbf{u}) = \alpha\mathbf{w}.$$

Therefore $\mathbf{w} + \mathbf{x} \in T(V)$ and $\alpha\mathbf{w} \in T(V)$, so $T(V)$ is a subspace of W. □

Theorem 2. *A linear transformation from V to W is determined by where it sends any spanning set of V. That is, if $\mathbf{v}_1, \ldots, \mathbf{v}_n \in V$ are such that $V = Span\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$, and $T_1 : V \to W$ and $T_2 : V \to W$ are two linear transformations such that $T_1(\mathbf{v}_i) = T_2(\mathbf{v}_i)$ for each i, then $T_1 = T_2$.*

Proof. Let $\mathbf{v} \in V$. Then we can write

$$\mathbf{v} = \sum_{i=1}^{n} a_i \mathbf{v}_i$$

for some $a_1, \ldots, a_n \in \mathbb{R}$. Then

$$\begin{aligned} T_1(\mathbf{v}) &= T_1\left(\sum_{i=1}^{n} a_i \mathbf{v}_i\right) \\ &= \sum_{i=1}^{n} a_i T_1(\mathbf{v}_i) \\ &= \sum_{i=1}^{n} a_i T_2(\mathbf{v}_i) \\ &= T_2\left(\sum_{i=1}^{n} a_i \mathbf{v}_i\right) \\ &= T_2(\mathbf{v}). \end{aligned}$$

Therefore $T_1 = T_2$. □