MATH 54 Final Exam (Special Version)

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Each problem is worth 30 points. Please show your work, except for problem 1, where only the answer is necessary.

Name:

Student ID:

Total	
1	
2	
3	
4	
5	

1. Answer the following statements with "true," "false," or "I don't know." Five points will be given for the correct answer, and three points will be given for each "I don't know" answer.

   (a) The function

   $$f(x) = \begin{cases} \frac{1}{x-1} & x > 1 \\ -4 & x = 1 \\ -8 & x < 1 \end{cases}$$

   is piecewise continuous on the interval $[-2, 2]$.

   (b) Let A be an $n \times n$ matrix such that $A^2 - 4A + 4I = 0$. Then A is diagonalizable.

   (c) Let U and V both be subspaces of a linear vector space W. Then the union of U and V, $U \cup V$, is also a subspace of W.

(d) Define

$$f(x) = x \sin x \tag{1}$$

and

$$g(x) = e^x \cos x. \tag{2}$$

Then the functions f and g are linearly independent on the interval $(-\pi, \pi)$.

(e) There exist functions $p(x)$ and $q(x)$ which are continuous on the interval $(-\pi, \pi)$ such that f and g, as defined in (1) and (2), are both solutions to the differential equation $y'' + p(x)y' + q(x)y = 0$ on the interval $(-\pi, \pi)$.

(f) Let A be a 3×3 matrix. Let $h(t) = \det(\exp(At))$. Then $h(t) \neq 0$ for all $t \in \mathbb{R}$.

2. Consider the following system of differential equations.

$$\mathbf{x}'(t) = \frac{1}{3}\begin{pmatrix} 6 & 0 & 0 \\ -5 & -4 & -5 \\ 10 & 2 & -2 \end{pmatrix}\mathbf{x}(t) \tag{3}$$

(a) Give the general solution to (3).

(b) Give the solution to (3) satisfying the initial condition

$$\mathbf{x}(0) = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}.$$

3. Consider the following differential equation.

$$y''' - 3y'' + 4y = 0 \tag{4}$$

(a) Convert (4) into a system of first-order linear differential equations.

(b) Give the general solution to the system of differential equations which was your answer to part (a).

(c) Give the general solution to (4).

(d) Give the solution to (4) satisfying the initial conditions $y(0) = 0$, $y'(0) = 0$, and $y''(0) = -1$.

4. Let P_2 have the following inner product.

$$\langle p, q \rangle = p(-2)q(-2) + p(0)q(0) + p(1)q(1) \tag{5}$$

(a) Give an orthogonal basis for

$$S = \{p \in P_2 \mid p(0) = p(1)\}$$

with respect to the inner product (5).

(b) Let $T : P_2 \to P_2$ be the linear transformation which orthogonally projects polynomials onto S with respect to the inner product (5). Give the matrix for T with respect to the ordered basis $\{x^2 - x, x + 1, x - 1\}$.

5. Consider the following partial differential equation.

$$u_{xx} - u_t - tu_t + u = 0 \tag{6}$$

(a) Transform (6) into a collection of ordinary differential equations using separation of variables.

(b) Using your answer to part (a), find all solutions to (6) satisfying the boundary conditions

$$u(0,t) = u(\pi,t) = 0.$$

(c) Find $u(x, t)$ satisfying the following conditions.

$$\begin{aligned}
u_{xx} - u_t - tu_t + u &= 0 \\
u(0, t) &= 0 \\
u(\pi, t) &= 0 \\
u(x, 0) &= \begin{cases} x^2 & 0 \leq x \leq \frac{\pi}{2} \\ (x - \pi)^2 & \frac{\pi}{2} < x \leq \pi \end{cases}
\end{aligned}$$