Theorem 1. *Let A be an $n \times n$ matrix, and let $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ be (nonzero) eigenvectors for A with distinct eigenvalues. Then these vectors are linearly independent.*

Proof. Assume they're linearly dependent. By reordering the $\mathbf{v}_i$'s if necessary, there exists some k such that $1 \leq k \leq n$ and coefficients $a_1, a_2, \ldots, a_k$ such that

$$\mathbf{0} = \sum_{i=1}^{k} a_i \mathbf{v}_i$$

and $a_i \neq 0$ for all i. Let λ_i be the eigenvalue associated to $\mathbf{v}_i$ for each i. Then

$$\begin{aligned}
\mathbf{0} &= (A - \lambda_k I)\mathbf{0} \\
&= (A - \lambda_k I) \sum_{i=1}^{k} a_i \mathbf{v}_i \\
&= \sum_{i=1}^{k} a_i (A - \lambda_k I)\mathbf{v}_i \\
&= \sum_{i=1}^{k} a_i (A\mathbf{v}_i - \lambda_k I \mathbf{v}_i) \\
&= \sum_{i=1}^{k} a_i (\lambda_i \mathbf{v}_i - \lambda_k \mathbf{v}_i) \\
&= \sum_{i=1}^{k} a_i (\lambda_i - \lambda_k)\mathbf{v}_i \\
&= \sum_{i=1}^{k-1} a_i (\lambda_i - \lambda_k)\mathbf{v}_i.
\end{aligned}$$

For all $i < k$, $\lambda_i - \lambda_k \neq 0$, since the $\mathbf{v}_i$'s were presumed to have distinct eigenvalues. Therefore $a_i(\lambda_i - \lambda_k) \neq 0$ for all $i < k$.

Now we have obtained a linear dependence relation

$$\mathbf{0} = \sum_{i=1}^{k-1} a_i (\lambda_i - \lambda_k)\mathbf{v}_i$$

involving only the vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_{k-1}$ where *all* of the coefficients are nonzero. We can repeat this process (the next step is to multiply by the matrix $A - \lambda_{k-1} I$, etc.) until we arrive at an equation

$$\mathbf{0} = \alpha \cdot \mathbf{v}_1$$

for some $\alpha \neq 0$. This implies $\mathbf{v}_1 = \mathbf{0}$, which is a contradiction.

Therefore $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ are linearly independent. □