Commutativity and Matrix Multiplication

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We say that two $n \times n$ matrices A and B *commute* if $AB = BA$. This is only the case in certain special circumstances. For certain 2×2 matrices, we can compute the set of all matrices with which that matrix commutes.

- Let $A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$. Suppose a matrix $E = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ commutes with A. Then

$$\begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} = AE = EA = \begin{pmatrix} a & 0 \\ c & 0 \end{pmatrix},$$

  which is equivalent to the condition $b = c = 0$.

- Let $B = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. Suppose a matrix $E = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ commutes with B. Then

$$\begin{pmatrix} c & d \\ -a & -b \end{pmatrix} = BE = EB = \begin{pmatrix} -b & a \\ -d & c \end{pmatrix},$$

  which is equivalent to the conditions $a = d$ and $b = -c$.

- Let $D = \begin{pmatrix} 2 & 3 \\ 0 & 5 \end{pmatrix}$. Suppose a matrix $E = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ commutes with D. Then

$$\begin{pmatrix} 2a+3c & 2b+3d \\ 5c & 5d \end{pmatrix} = DE = ED = \begin{pmatrix} 2a & 3a+5b \\ 2c & 3c+5d \end{pmatrix},$$

  which is equivalent to the conditions $2a+3c = 2a$, $2c = 5c$, $2b+3d = 3a+5b$, and $5d = 3c+5d$. These conditions are in turn equivalent to $c = 0$ and $b = d - a$.

Now suppose there is a matrix $E = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ which commutes with every 2×2 matrix. Since it commutes with A from above, $b = c = 0$. Then since E commutes with B from above, $a = d$. Therefore, $E = \lambda I$ for some $\lambda \in \mathbb{R}$.

We will show that λI_n commutes with any $n \times n$ matrix for any $\lambda \in \mathbb{R}$. Let $F = (f_{ij})$ be an $n \times n$ matrix. Let $\lambda I_n = (l_{ij})$. Note that $l_{ij} = \lambda$ if $i = j$ and $l_{ij} = 0$ if $i \neq j$. Then the (i, j)-th entry of $F(\lambda I_n)$ is given by

$$\sum_{k=1}^{n} f_{ik} l_{kj} = \lambda f_{ij},$$

and the (i, j)-th entry of $(\lambda I_n)F$ is given by

$$\sum_{k=1}^{n} l_{ik} f_{kj} = \lambda f_{ij}.$$

Therefore F commutes with λI_n.