Theorem 1. *Let A and B be $n \times n$ lower triangular matrices. Then AB is also lower triangular.*

Proof. Let $A = (a_{ij})$ and $B = (b_{ij})$. Denote the (i,j)-th entry of AB by c_{ij}. Then we have that $a_{ij} = b_{ij} = 0$ whenever $i < j$, and we want to show that $c_{ij} = 0$ whenever $i < j$. By the definition of matrix mulitplication,

$$c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}. \tag{1}$$

Let $i < j$, and let k be any integer between 1 and n. If $i \geq k$ and $k \geq j$, then $i \geq j$, which is false by assumption. Therefore either $i < k$ or $k < j$. If $i < k$ then $a_{ik} = 0$, and if $k < j$ then $b_{kj} = 0$, so in either case, $a_{ik} b_{kj} = 0$. Therefore, whenever $i < j$, every term in the sum on the right hand side of (1) is equal to 0, so $c_{ij} = 0$. Therefore AB is lower triangular. □

Theorem 2. *Let A and B be $n \times n$ unit lower triangular matrices. Then AB is also unit lower triangular.*

Proof. Theorem 1 tells us that AB is lower triangular, so we must only show that all of its diagonal entries are equal to 1. As before, let $A = (a_{ij})$ and $B = (b_{ij})$. Then a_{ij} and b_{ij} are both 0 if $i < j$ and 1 if $i = j$. Let c_i denote the i-th diagonal entry of AB. By the definition of matrix multiplication,

$$c_i = \sum_{k=1}^{n} a_{ik} b_{ki}. \tag{2}$$

The only k for which it is true that both $i \geq k$ and $k \geq i$ is $k = i$. Therefore, the only nonzero term in the sum on the right hand side of (2) is for when $k = i$. This term is $a_{ii} b_{ii}$, which is equal to 1. Therefore $c_i = 1$ for all i, so AB is unit lower triangular. □