Sequences and Series Review

Edward Carter

October 30, 2005

1 Things to Know

Items marked with a star (*) are important enough to include in this list, but you probably won't need to know them for the midterm.

- What the contrapositive, inverse, and converse of a statement are. Which one of those is logically equivalent to the original statement?
- $\lim_{n\to\infty}\left(1+\frac{r}{n}\right)^n = e^r$
- (*) A sequence $\{a_n\}$ has limit L and we write

  $$\lim_{n\to\infty} a_n = L$$

  if for every $\epsilon > 0$ there exists N such that $|a_n - L| < \epsilon$ whenever $n > N$.
- (*) A sequence $\{a_n\}$ diverges to ∞ and we write

  $$\lim_{n\to\infty} a_n = \infty$$

  if for every positive number M there exists N such that $a_n > M$ whenever $n > N$. The definition of $\{a_n\}$ diverging to $-\infty$ is analogous.
- If $\lim_{x\to\infty} f(x) = L$ and $f(n) = a_n$ when n is an integer, then $\lim_{n\to\infty} a_n = L$.
- If $\{a_n\}$ and $\{b_n\}$ are convergent sequences and c is a constant, then
  - $\lim_{n\to\infty}(a_n + b_n) = \lim_{n\to\infty} a_n + \lim_{n\to\infty} b_n$
  - $\lim_{n\to\infty}(a_n - b_n) = \lim_{n\to\infty} a_n - \lim_{n\to\infty} b_n$
  - $\lim_{n\to\infty} ca_n = c \cdot \lim_{n\to\infty} a_n$
  - $\lim_{n\to\infty} c = c$
  - $\lim_{n\to\infty}(a_n b_n) = (\lim_{n\to\infty} a_n)(\lim_{n\to\infty} b_n)$
  - $\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{\lim_{n\to\infty} a_n}{\lim_{n\to\infty} b_n}$ if $\lim_{n\to\infty} b_n \neq 0$
  - $\lim_{n\to\infty} a_n^p = (\lim_{n\to\infty} a_n)^p$ if $a_n > 0$ and $p > 0$
- Squeeze Theorem: If $a_n \leq b_n \leq c_n$ for $n \geq n_0$ and $\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = L$, then $\lim_{n\to\infty} b_n = L$.
- $\lim_{n\to\infty} |a_n| = 0$ if and only if $\lim_{n\to\infty} a_n = 0$.
- If a sequence is bounded and monotonic, then it is convergent.
- We define $\sum_{n=1}^{\infty} a_n$ to be $\lim_{k\to\infty} \sum_{n=1}^{k} a_n$.

- What a geometric series is, and when it converges. Does it converge absolutely?
- If the series $\sum_{n=1}^{\infty} a_n$ is convergent, then $\lim_{n\to\infty} a_n = 0$.
- If $\sum a_n$ and $\sum b_n$ are convergent series and c is a constant, then the series $\sum ca_n$, $\sum(a_n + b_n)$ and $\sum(a_n - b_n)$ also converge, and
  - $\sum_{n=1}^{\infty} ca_n = c\sum_{n=1}^{\infty} a_n$
  - $\sum_{n=1}^{\infty}(a_n + b_n) = \sum_{n=1}^{\infty} a_n + \sum_{n=1}^{\infty} b_n$
  - $\sum_{n=1}^{\infty}(a_n - b_n) = \sum_{n=1}^{\infty} a_n - \sum_{n=1}^{\infty} b_n$
- How to compute the limits of telescoping series. See page 717.
- The Integral Test: Suppose f is continuous on $[1, \infty)$, and positive and decreasing on $[n_0, \infty)$ for some n_0. Then let $a_n = f(n)$. Then the series $\sum_{n=1}^{\infty} a_n$ converges if and only if the integral $\int_1^{\infty} f(x)\,dx$ converges.
- The p-test: The p-series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ is convergent if and only if $p > 1$.
- (*) General remainder formulas. See page 727.
- The Comparison Test: Suppose that $\sum a_n$ and $\sum b_n$ are series with positive terms, and let n_0 be any positive integer. If $\sum b_n$ is convergent and $a_n \leq b_n$ for all $n \geq n_0$, then $\sum a_n$ is also convergent. If $\sum b_n$ is divergent and $a_n \geq b_n$ for all $n \geq n_0$, then $\sum a_n$ is also divergent.
- The Limit Comparison Test: Suppose that $\sum a_n$ and $\sum b_n$ are series with positive terms. If

  $$\lim_{n\to\infty} \frac{a_n}{b_n} = c$$

  where c is a finite number and $c > 0$, then either both series converge or both diverge.

  If you are trying to see whether some series converges or diverges and you get either 0 or ∞ as the limit, pick a different series to use for the comparison.
- The Alternating Series Test: Let b_n be a positive, decreasing sequence such that $\lim_{n\to\infty} b_n = 0$. Then the series $\sum_{n=1}^{\infty}(-1)^n b_n$ converges.
- (*) Alternating series remainder formulas. See page 738.
- A series $\sum a_n$ is called absolutely convergent if the series $\sum |a_n|$ is convergent. A series that is convergent but not absolutely convergent is called conditionally convergent.
- Any absolutely convergent series is convergent.
- The Ratio Test: For a series $\sum_{n=1}^{\infty} a_n$, compute $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$. If the limit does not exist or is equal to 1, the test is inconclusive. If the limit is less than 1, the series converges absolutely. If the limit is greater than one (or infinite), then the series diverges.
- The Root Test: For a series $\sum_{n=1}^{\infty} a_n$, compute $\lim_{n\to\infty} \sqrt[n]{|a_n|}$. If the limit does not exist or is equal to 1, the test is inconclusive. If the limit is less than 1, the series converges absolutely. If the limit is greater than one (or infinite), then the series diverges.
- (*) You can change the sum of a conditionally convergent series by rearranging the terms. This is not true of an absolutely convergent series.
- A power series centered at a is a series of the form $\sum_{n=0}^{\infty} c_n(x-a)^n$.

- Know how to compute the interval of convergence of a power series.
- A power series can be differentiated or integrated term-by-term inside its interval of convergence.
- If f has a power series representation at a, then

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)(x-a)^n}{n!}.$$

  Note that $f^{(0)}(a) = f(a)$.
- Standard power series expansions:
  - $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$
  - $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$
  - $\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$
  - $\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$
  - (*) $\arctan x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1}$
  - $(1+x)^k = \sum_{n=0}^{\infty} \binom{k}{n} x^k$, where $\binom{k}{n} = \frac{k(k-1)(k-2)\cdots(k-n+1)}{n!}$ if $n \geq 1$ and $\binom{k}{0} = 1$
- Multiplication and division of power series.

2 Problems

Look at the Chapter 11 Review (page 786) and Chapter 11 Problems Plus (page 789). Try to do all the problems there. The problems below come from *Berkeley Problems in Mathematics*.

1. Which of the following series converge?
   (a) $\sum_{n=1}^{\infty} \frac{(2n)!(3n)!}{n!(4n)!}$
   (b) $\sum_{n=1}^{\infty} \frac{1}{n^{1+1/n}}$
2. Let $a_1, a_2, a_3, \ldots$ be positive numbers.
   (a) Prove that $\sum \sqrt{a_n a_{n+1}}$ converges if $\sum a_n$ converges.
   (b) Prove that the converse of the above statement is false.
3. For each $a, b, c \in \mathbb{R}$, consider the series

$$\sum_{n=3}^{\infty} \frac{a^n}{n^b (\ln n)^c}.$$

   Determine the values of a, b, and c for which the series
   (a) converges absolutely;
   (b) converges conditionally;
   (c) diverges.

4. For which real numbers x does the infinite series

$$\sum_{n=1}^{\infty} \frac{\sqrt{n+1} - \sqrt{n}}{n^x}$$

converge?

5. For which values of a does the series

$$\sum_{n=1}^{\infty} \left(\frac{1}{n} - \sin\left(\frac{1}{n}\right) \right)^a$$

converge?

6. Let A be the set of positive integers that do not contain the digit 9 in their decimal expansions. Prove that the series

$$\sum_{a \in A} \frac{1}{a} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \frac{1}{10} + \frac{1}{11} + \cdots$$

converges.