1. (3pts) Determine (and justify!) whether the sequence is convergent and divergent, and give the limit if it converges.

$$a_n = \frac{n + \cos(\frac{1}{n})}{n\cos(\pi n) + \sin(\frac{1}{n})}$$

If n is even, then

$$a_n = \frac{n + \cos\left(\frac{1}{n}\right)}{n\cos(\pi n) + \sin\left(\frac{1}{n}\right)} = \frac{n + \cos\left(\frac{1}{n}\right)}{n + \sin\left(\frac{1}{n}\right)},$$

so that

$$\frac{n}{n+1} \leq a_n \leq \frac{n+1}{n}.$$

By the squeeze theorem, this subsequence approaches 1.

If n is odd, then

$$a_n = \frac{n + \cos\left(\frac{1}{n}\right)}{n\cos(\pi n) + \sin\left(\frac{1}{n}\right)} = -\frac{n + \cos\left(\frac{1}{n}\right)}{n - \sin\left(\frac{1}{n}\right)},$$

so that a_n is negative for all sufficiently large n. Therefore this subsequence cannot possibly converge to 1. Since the original a_n has a subsequence that converges to 1 and another subsequence that doesn't converge to 1, a_n diverges.

2. (3pts) Determine whether the series $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n \ln n}$ is convergent or divergent.

Let $f(x) = \frac{1}{x \ln x}$. Since the functions $\frac{1}{x}$ and $\frac{1}{\ln x}$ are both continuous, positive, and decreasing for $x > 1$, f is continuous, positive, and decreasing for $x > 1$. Now we can use the integral test. With $u = \ln x$,

$$\int_2^{\infty} \frac{dx}{x \ln x} = \int_{\ln 2}^{\infty} \frac{du}{u}$$

which diverges by the p-test. Therefore the series diverges as well.

3. (4pts) Determine whether the statement true or false. If false give a counterexample.

   (a) (2pts) $\lim\limits_{n\to\infty} a_n = 1$, then $\lim\limits_{n\to\infty} a_{n+100} = 1$

   True. The sequence a_{n+100} is a subsequence of a_n, so since a_n converges to 1, a_{n+100} does as well.

   (b) (2pts) $\lim\limits_{n\to\infty} a_n = 0$, then $\displaystyle\sum_{n=1}^{\infty} a_n$ converges.

   False. Let $a_n = \frac{1}{n}$.