1. (3pts) Determine (and justify!) whether the sequence is convergent and divergent, and give the limit if it converges.

$$a_n = \frac{(1+n)\left(1+n^2\right)}{\cos(n)+n^3}$$

We have that

$$a_n = \frac{(1+n)\left(1+n^2\right)}{\cos(n)+n^3} = \frac{n^3+n^2+n+1}{n^3+\cos(n)} = \frac{1+\frac{1}{n}+\frac{1}{n^2}+\frac{1}{n^3}}{1+\frac{\cos(n)}{n^3}}.$$

So, if we let $b_n = 1+\frac{1}{n}+\frac{1}{n^2}+\frac{1}{n^3}$ and $c_n = 1+\frac{\cos(n)}{n^3}$, then $\lim_{n\to\infty} b_n = \lim_{n\to\infty} c_n = 1$. Since $a_n = \frac{b_n}{c_n}$, the sequence converges, and

$$\lim_{n\to\infty} a_n = \frac{\lim_{n\to\infty} b_n}{\lim_{n\to\infty} c_n} = 1.$$

2. (3pts) Determine whether the series $\displaystyle\sum_{n=1}^{\infty} ne^{-n^2}$ is convergent or divergent.

Let $f(x) = xe^{-x^2}$. Then f is clearly continuous for all x and positive for all $x > 0$. We have that

$$f'(x) = e^{-x^2} - 2x^2 = e^{-x^2}(1-2x^2),$$

which is negative if $x \geq 1$. Therefore we can apply the integral test.

Letting $u = x^2$, we have

$$\int_1^{\infty} xe^{-x^2}\,dx = \frac{1}{2}\int_1^{\infty} e^{-u}\,du = \frac{1}{2e}.$$

Since the integral converges, the series converges as well.

3. (4pts) Determine whether the statement true or false. If false give a counterexample.

   (a) (2pts) The sequence $\{a_n\}$ diverges, the sequence $\{b_n\}$ has the limt zero then $\{a_nb_n\}$ diverges.
   False. Let $a_n = n$ and $b_n = \frac{1}{n^2}$. Then $a_nb_n = \frac{1}{n}$, which converges.

   (b) (2pts) The sequence a_n converges. The sequence b_n is monotonically increasing and $b_n \leq a_n$. Then the sequence $\{b_n\}$ converges.

   True. Since a_n converges, it is bounded above by some constant. Then b_n is bounded above by this same constant. Since b_n is monotonically increasing and bounded above, it must converge.