1. (3pts) Evaluate the integral $\displaystyle\int \frac{\sin x}{-\cos x + \sin x + 1}\, dx$.

   Letting $t = \tan\left(\frac{x}{2}\right)$, we have

$$\begin{aligned}\int \frac{\sin x}{-\cos x + \sin x + 1}\, dx &= \int \frac{\frac{4t\ dt}{(1+t^2)^2}}{\frac{t^2-1}{1+t^2} + \frac{2t}{1+t^2} + \frac{1+t^2}{1+t^2}} \\ &= \int \frac{4t\ dt}{(1+t^2)(2t^2+2t)} \\ &= \int \frac{2\ dt}{(1+t^2)(1+t)}.\end{aligned}$$

   We know that for some A, B, C,

$$\frac{2}{(1+t^2)(1+t)} = \frac{A}{1+t} + \frac{Bt+C}{1+t^2}.$$

   Multiplying both sides by $1+t$ and setting $t = -1$, we have that $A = 1$. Then

$$\frac{Bt+C}{1+t^2} = \frac{2}{(1+t^2)(1+t)} - \frac{1+t^2}{(1+t^2)(1+t)} = \frac{1-t^2}{(1+t^2)(1+t)} = \frac{1-t}{1+t^2}.$$

   So, we have that

$$\begin{aligned}\int \frac{2\ dt}{(1+t^2)(1+t)} &= \int \frac{dt}{1+t} + \int \frac{dt}{1+t^2} - \int \frac{t\ dt}{1+t^2} \\ &= \ln|1+t| + \arctan t - \frac{1}{2}\ln|1+t^2| + C.\end{aligned}$$

   Since $\frac{1}{\sqrt{1+t^2}} = \frac{1}{\sqrt{1+\tan^2\left(\frac{x}{2}\right)}} = \cos\left(\frac{x}{2}\right)$,

$$\int \frac{\sin x}{-\cos x + \sin x + 1}\, dx = \ln\left|\cos\left(\frac{x}{2}\right) + \sin\left(\frac{x}{2}\right)\right| + \frac{x}{2} + C.$$

2. (3pts) Evaluate the integral: $\displaystyle\int \frac{3\ dx}{x^2 - x - 2}$.

   Since $x^2 - x - 2 = (x-2)(x+1)$, there exist A, B such that

$$\frac{3}{x^2-x-2} = \frac{A}{x-2} + \frac{B}{x+1}.$$

   Multiplying both sides by $x-2$ and setting $x = 2$, we obtain $A = 1$. Multiplying both sides by $x+1$ and setting $x = -1$, we obtain $B = -1$. So,

$$\begin{aligned}\int \frac{3\ dx}{x^2-x-2} &= \int \frac{dx}{x-2} - \int \frac{dx}{x+1} \\ &= \ln|x-2| - \ln|x+1| + C.\end{aligned}$$

3. (4pts) Let $F(x)$ be defined by $F(x) = \int_0^x \tan^{-1} t \, dt$. If we apply the midpoint rule to approximate the integral $\int_{-1/2}^{1/2} F(x)\, dx$, how many subintervals do we need in order for the error to be at most 1/2400?

By the Fundamental Theorem of Calculus, $F'(x) = \arctan x$. Then

$$|F''(x)| = \left|\frac{1}{1+x^2}\right| = \frac{1}{1+x^2}.$$

This function reaches its maximum value of 1 at $x = 0$. So, let $K = 1$.

We want n to be such that

$$|E_M| \leq \frac{K(b-a)^3}{24n^2} = \frac{1}{24n^2} \leq \frac{1}{2400}.$$

This is equivalent to $n^2 \geq 100$ or $n \geq 10$.