Math 1B, Quiz 12 Solutions Section 211 GSI Carter

1. (4pts) Solve using variation of parameters: $y'' - 2y' - 3y = e^x$.

   The characteristic equation is $0 = r^2 - 2r - 3 = (r-3)(r+1)$, so let $y_1 = e^{3x}$ and $y_2 = e^{-x}$. Then

$$u_1' e^{3x} + u_2' e^{-x} = 0$$
$$3u_1' e^{3x} - u_2' e^{-x} = e^x.$$

   Adding these two equations, we obtain $4u_1' e^{3x} = e^x$, or $u_1' = \frac{1}{4}e^{-2x}$. Therefore $u_1 = -\frac{1}{8}e^{-2x} + c_1$. Since $u_2' = -u_1' e^{4x}$, $u_2' = -\frac{1}{4}e^{2x}$. Therefore $u_2 = -\frac{1}{8}e^{2x} + c_2$. So, the general solution to the differential equation is

$$y = u_1 y_1 + u_2 y_2 = c_1 e^{3x} + c_2 e^{-x} - \frac{1}{8}e^x - \frac{1}{8}e^x = c_1 e^{3x} + c_2 e^{-x} - \frac{1}{4}e^x.$$

2. (6pts) Solve using any technique: $y'' - 2xy' - y = 0$.

   Let $y = \sum_{n=0}^{\infty} c_n x^n$. Then $y' = \sum_{n=1} n c_n x^{n-1}$ and

$$y'' = \sum_{n=2} n(n-1) c_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1) c_{n+2} x^n.$$

   Then $xy' = \sum_{n=0} n c_n x^n$, so

$$0 = y'' - 2xy' - y = \sum_{n=0}^{\infty} \left[(n+2)(n+1)c_{n+2} - 2nc_n - c_n\right] x^n.$$

   Therefore, for each $n \geq 0$,

$$c_{n+2} = \frac{(2n+1)c_n}{(n+2)(n+1)}.$$

   Therefore

$$c_{2k} = \frac{(4k-3)(4k-5)\cdots(1)c_0}{(2k)!}$$

   and

$$c_{2k+1} = \frac{(4k-1)(4k-3)\cdots(1)c_1}{(2k+1)!}$$

   So, the general solution to the differential equation is

$$y = c_0 \sum_{n=0}^{\infty} \frac{(4n-3)(4n-5)\cdots(1)x^{2n}}{(2n)!} + c_1 \sum_{n=0}^{\infty} \frac{(4n-1)(4n-3)\cdots(1)x^{2n+1}}{(2n+1)!}.$$