

1. (3pts) Solve the boundary-value problem $4y'' + 4y' + y = 0$, $y(0) = 1$, $y(1) = 0$.

The characteristic equation is $0 = 4r^2 + 4r + 1 = (2r + 1)^2$, so the general solution is $y = c_1 e^{-x/2} + c_2 x e^{-x/2}$. Since $y(0) = 1$, $c_1 = 1$, so the general solution reduces to $y = e^{-x/2} + c_2 x e^{-x/2}$. Since $y(1) = 0$, $0 = e^{-1/2} + c_2 e^{-1/2}$, so that $c_2 = -1$. Therefore, the solution to the boundary-value problem is

$$y = e^{-x/2} - x e^{-x/2}.$$

2. (3pts) Use the method of undetermined coefficients to find the general solution for $4y'' + 4y' + y = \cos x$.

As in the previous problem, the general solution to the homogeneous equation is $y_c = c_1 e^{-x/2} + c_2 x e^{-x/2}$. Let $y_p = A \cos x + B \sin x$. Then $y_p' = -A \sin x + B \cos x$ and $y_p'' = -A \cos x - B \sin x$, so

$$4y_p'' + 4y_p' + y_p = (\cos x)(A + 4B - 4A) + (\sin x)(B - 4A - 4B) = \cos x.$$

Therefore, $-3A + 4B = 1$ and $-4A - 3B = 0$, so $A = -\frac{3}{25}$ and $B = \frac{4}{25}$. So, the general solution for the differential equation is

$$y = y_p + y_c = c_1 e^{-x/2} + c_2 x e^{-x/2} + \frac{4}{25} \sin x - \frac{3}{25} \cos x.$$

3. (4pts) Use variation of parameters to find the general solution for $y'' + y = \frac{1}{\cos x}$.

The characteristic equation is $0 = r^2 + 1 = (r + i)(r - i)$, so $\alpha = 0$ and $\beta = 1$, and the general solution for the homogeneous equation is $y_c = c_1 \cos x + c_2 \sin x$. So, let $y_1 = \cos x$ and $y_2 = \sin x$. Then

$$\begin{aligned} 0 &= u_1' \cos x + u_2' \sin x = u_1' \sin x \cos x + u_2' \sin^2 x \\ \frac{1}{\cos x} &= -u_1' \sin x + u_2' \cos x. \end{aligned}$$

Then since $1 = -u_1' \sin x \cos x + u_2' \cos^2 x$, $u_2' = 1$. Thus, $u_2 = x + c_2$. Then

$$\begin{aligned} u_1 &= \int (-u_2' \tan x) dx \\ &= \int (-\tan x) dx \\ &= \ln |\cos x| + c_1. \end{aligned}$$

So, the general solution is

$$y = u_1 y_1 + u_2 y_2 = c_1 \cos x + c_2 \sin x + (\cos x) \ln |\cos x| + x \sin x.$$