1. (3pts) Give the general solution for the differential equation $y' + e^x y = e^x$.

$$I(x) = e^{\int e^x \, dx} = e^{e^x}$$

$$\begin{aligned} e^{e^x} y' + e^{e^x} e^x y &= e^{e^x} e^x \\ e^{e^x} y &= e^{e^x} + C \\ y &= 1 + Ce^{-e^x} \end{aligned}$$

2. (4pts) Solve the initial-value problem $y' - (\cot x)y = 1$, $y(\frac{\pi}{2}) = 1$.

$$\begin{aligned} I(x) = e^{-\int \cot x \, dx} &= e^{-\int \frac{\cos x}{\sin x} \, dx} \\ &= e^{-\int \frac{du}{u}} \qquad \text{(with } u = \sin x\text{)} \\ &= e^{-\ln u} \\ &= \frac{1}{u} \\ &= \csc x \end{aligned}$$

$$\begin{aligned} (\csc x)y' - (\csc x \cot x)y &= \csc x \\ (\csc x)y &= \int \csc x \, dx \\ &= \ln|\csc x + \cot x| + C \end{aligned}$$

Since $y\left(\frac{\pi}{2}\right) = 1$, we have that $1 = C$. Therefore,

$$y = (\ln|\csc x + \cot x| + 1)\sin x.$$

3. (3pts) Solve the following initial-value problem.

$$\begin{aligned} y' &= \frac{x^2 + y^2}{2xy} \\ y(1) &= 0 \end{aligned}$$

If $v = \frac{y}{x}$, then $\frac{x^2+y^2}{2xy} = \frac{1}{2v} + \frac{v}{2}$, and $\frac{dv}{dx} = \frac{\frac{1}{2v} + \frac{v}{2} - v}{x} = \frac{1-v^2}{2vx}$. So,

$$\begin{aligned} \int \frac{2v}{1 - v^2} \, dv &= \frac{dx}{x} \\ -\ln|1 - v^2| &= \ln|x| + C \\ 1 - v^2 &= \frac{A}{x} \\ v^2 &= 1 - \frac{A}{x} \\ y^2 &= x^2 - Ax \\ y &= \pm\sqrt{x^2 - Ax}. \end{aligned}$$

Since $y(1) = 0$, $0 = \sqrt{1 - A}$, so that $A = 1$. Therefore,

$$y = \pm\sqrt{x^2 - x}.$$