MATH 279 HOMEWORK 5

1. Let m be an order function on $\mathbb{R}^{2n}$ and assume that $a \in S(m)$ is globally elliptic, that is there exists a constant $c > 0$ such that

$$|a(x,\xi;h)| \geq cm(x,\xi) \quad \text{for all} \quad (x,\xi) \in \mathbb{R}^{2n}, \quad h \in (0,1].$$

(a) Show that $a^{-1} \in S(m^{-1})$.

(b) Arguing by induction construct symbols $b_0 := a^{-1}, b_1, b_2, \dots$ and $r_1, r_2 \dots$ such that $b_j \in h^j S(m^{-1})$, $r_j \in h^j S(1)$, and for all $j \geq 0$

$$a \# (b_0 + \dots + b_j) = 1 + r_{j+1}.$$

(c) Using Borel's Lemma, take an asymptotic sum

$$b \sim \sum_{j=0}^{\infty} b_j \in S(m^{-1})$$

and show that $a \# b = 1 + \mathcal{O}(h^\infty)_{S(1)}$.

(d) Arguing similarly, we can construct $\tilde{b} \in S(m^{-1})$ such that $\tilde{b} \# a = 1 + \mathcal{O}(h^\infty)_{S(1)}$. Using associativity of the # product, show that $\tilde{b} = b + \mathcal{O}(h^\infty)_{S(m^{-1})}$ and thus

$$a^{\mathrm{w}} b^{\mathrm{w}} = I + \mathcal{O}(h^\infty)_{L^2 \to L^2}, \quad b^{\mathrm{w}} a^{\mathrm{w}} = I + \mathcal{O}(h^\infty)_{L^2 \to L^2}.$$

2. Let m be an order function and $a \in S(m)$ be nonnegative and globally elliptic.

(a) Show that $\sqrt{a} \in S(\sqrt{m})$.

(b) Arguing as in the previous exercise, construct $b \sim \sum_{j=0}^{\infty} b_j \in S(\sqrt{m})$, $b_j \in h^j S(\sqrt{m})$, $b_0 = \sqrt{a}$, such that $b \# b = a + \mathcal{O}(h^\infty)_{S(m)}$.

(c) Assume now that $m \equiv 1$. Writing

$$a^{\mathrm{w}} = (b^{\mathrm{w}})^2 + \mathcal{O}(h^\infty)_{L^2 \to L^2}$$

show the easy Gårding inequality: there exists $c > 0$, $h_0 > 0$ such that for all $h \in (0, h_0)$

$$\langle a^{\mathrm{w}}(x, hD_x)u, u \rangle \geq c\|u\|_{L^2}^2 \quad \text{for all} \quad u \in L^2(\mathbb{R}^n).$$

(For this part one could actually just use $b := \sqrt{a}$.)