

252 Exercise XVIII

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1. (For fun-loving folks!) A group G is called funny if it has n irreducible \mathbb{C} -characters with degrees $1, 2, \dots, n$. Show that all funny groups are trivial.

Solution Let $a_n = |G| = \sum_{k=1}^n k^2 = 1/6n(n+1)(2n+1)$. We will show that there does not exist such a group for $n > 1$. Since \mathbb{C} is algebraically closed, so \mathbb{C} is splitting field of G , every prime less than n has to divide a_n . So, we have $\prod_{p < n} p \leq a_n$. By easy computations we know that for $1 < n < 14$ there is not such a group.

I will use induction on n .

When $n = 14$, $\prod_{p < n} p = 2 \times 3 \times 5 \times 7 \times 11 \times 13 = 30030 > 8 \times a_n = 8120 \geq a_{2n}$ since

$$a_{2n} = \frac{(2n)(2n+1)(4n+1)}{6} \leq \frac{(2n)(2n+2)(4n+2)}{6} = 8 \times a_n$$

By “Bertrand’s conjecture” (proved by Chebyshev), there is always a prime p between n and $2n$. Thus, for $n \geq 14$, we know that $\prod_{p < n} p > 8a_n > a_{2n}$ and $\prod_{p < 2n} p > n \prod_{p < n} p > na_{2n} > 8a_{2n}$. Since a_n is an increasing function on n , we have $\prod_{p < m} p \geq \prod_{p < n} p > a_{2n} > a_m$ for all $n < m \leq 2n$. By induction, we have that there does not exist such a group for all $n > 1$.

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