MATH 252 ASSIGNMENT 18

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(7) Let $G = \langle r, s\rangle$ be the semidehedram group of order 16 defined in Exercise IX.5. In the solution to IX.6, we remarked that the two faithful 2-dimensional $\mu_6, \mu_7 \in \operatorname{Irr}(G)$ both restrict to the unique 2-dimensional $\nu \in \operatorname{Irr}(H)$ where $H = \langle r^2, sr\rangle \cong Q_8$.

(a) What does this tell you about ν^G?

(b) Given this hindsight, recompute μ_6, μ_7 by starting with ν^G.

(c) Finally, use Exercise XVIII.6 to show that G is an M-group.

Solution.

(a) We show $\nu^G = \mu_6 + \mu_7$. We write the charater table of H, and we write the restricted characters of G below:

	1	r^4	r^2	sr	r^2sr
$\lvert h\rvert$	1	1	2	2	2
ν_1	1	1	1	1	1
ν_2	1	1	1	-1	-1
ν_3	1	1	-1	1	-1
ν_4	1	1	-1	-1	1
ν_5	2	-2	0	0	0
$(\mu_1)\vert_H$	1	1	1	1	1
$(\mu_2)\vert_H$	1	1	1	-1	-1
$(\mu_3)\vert_H$	1	1	1	-1	-1
$(\mu_4)\vert_H$	1	1	1	1	1
$(\mu_5)\vert_H$	2	2	-2	0	0
$(\mu_6)\vert_H$	2	-2	0	0	0
$(\mu_7)\vert_H$	2	-2	0	0	0

and note that $\nu = \nu_5$. It is clear then that the $(\mu_i)|_H$'s are given by:

$$\begin{pmatrix}\mu_1\\ \mu_2\\ \mu_3\\ \mu_4\\ \mu_5\\ \mu_6\\ \mu_7\end{pmatrix}_H = \begin{pmatrix}1&0&0&0&0\\ 0&1&0&0&0\\ 0&1&0&0&0\\ 1&0&0&0&0\\ 0&0&1&1&0\\ 0&0&0&0&1\\ 0&0&0&0&1\end{pmatrix}\begin{pmatrix}\nu_1\\ \nu_2\\ \nu_3\\ \nu_4\\ \nu_5\end{pmatrix}.$$

By the Frobenius Reciprocity Theorem, we have that

$$\begin{pmatrix}\nu_1\\ \nu_2\\ \nu_3\\ \nu_4\\ \nu_5\end{pmatrix}^G = \begin{pmatrix}1&0&0&1&0&0&0\\ 0&1&1&0&0&0&0\\ 0&0&0&0&1&0&0\\ 0&0&0&0&1&0&0\\ 0&0&0&0&0&1&1\end{pmatrix}\begin{pmatrix}\mu_1\\ \mu_2\\ \mu_3\\ \mu_4\\ \mu_5\\ \mu_6\\ \mu_7\end{pmatrix},$$

so $\nu_5^G = \mu_6 + \mu_7$.

(b) Suppose we knew $\nu_5^G = (4,-4,0,0,0,0,0)$. We know that $|\operatorname{Irr}(G)| = 7$ by counting conjugacy classes. The subgroup $N = Z(G) = \{1, r^4\} \lhd G$, so there are 4 linear characters $\mu_1, \dots, \mu_4$, and one irreducible 2-dimensional character μ_5 by looking at irreducible representaions of $G/N \cong D_4$. This gives us the following information in the character table C of G:

$$C = \begin{array}{c|ccccccc} & 1 & r^4 & r & r^2 & r^5 & s & sr \\ |g| & 1 & 1 & 2 & 2 & 2 & 4 & 4 \\ \hline \mu_1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ \mu_2 & 1 & 1 & 1 & 1 & 1 & -1 & -1 \\ \mu_3 & 1 & 1 & -1 & 1 & -1 & 1 & -1 \\ \mu_4 & 1 & 1 & -1 & 1 & -1 & -1 & 1 \\ \mu_5 & 2 & 2 & 0 & -2 & 0 & 0 & 0 \\ \mu_6 & a_1 & a_2 & a_3 & a_4 & a_5 & a_6 & a_7 \\ \mu_7 & b_1 & b_2 & b_3 & b_4 & b_5 & b_6 & b_7 \end{array}.$$

We must have that all remaining characters two dimensional by the Magic Equation.

<u>Claim:</u> $\nu_5^G = \mu_6 + \mu_7$.

Proof. We give two proofs as proving claims is fun.

<u>Proof 1:</u> Set

$$\nu_5^G = \sum_{i=1}^{7} c_i \mu_i \text{ where } c_i \in \mathbb{Z}_{\geq 0} \text{ for all } \quad i = 1, \dots, 7.$$

This immediately implies

$$c_1 + c_2 + c_3 + c_4 + 2c_5 + 2c_6 + 2c_7 = 4 \text{ and}$$
$$c_1 + c_2 + c_3 + c_4 + 2c_5 + c_6\mu_6(r^4) + c_7\mu_7(r^4) = -4.$$

Hence, at least one of c_6, c_7 is nonzero. Since G has nonreal conjugacy classes, Burnisde's Theorem implies μ_6 and μ_7 are nonreal, so $\overline{\mu_6} = \mu_7$. But we know ν_5^G is real by construction, so $c_6 = c_7$. The only way this can be true is if $c_1, \dots, c_5 = 0$ and $c_6 = c_7 = 1$, and $\mu_6(r^4) = \mu_7(r^4) = -2$.

<u>Proof 2:</u> By Frobenius Reciprocity, we have $[\nu_5^G, \mu_i]_G = [\nu_5, (\mu_i)_H]_H = 0$ for $i = 1, 2, 3, 4, 5$. Since μ_6, μ_7 are nonreal, $2\mu_6 \neq \nu_5^G \neq 2\mu_7$, so we must have $\nu_5^G = \mu_6 + \mu_7$. ☐

By the claim (and the proof of the claim),

$$(4,-4,0,0,0,0,0) = \nu_5^G = \mu_6 + \mu_7 = (2,-2,a_3,a_4,a_5,a_6,a_7) + (2,-2,\overline{a_3},\overline{a_4},\overline{a_5},\overline{a_6},\overline{a_7}),$$

so all the remaining a_i's must be purely imaginary! As μ_i is real on real conjugacy classes, we know $a_4 = a_6 = a_7 = 0$. Columns 3 and 5 are conjugate, so

$$\mu_6(g) = \overline{\mu_6(g^{-1})} = \mu_7(g^{-1}) = \overline{\mu_7(g)} \implies \begin{cases} \mu_6 = (2,-2,a_3,0,\overline{a_3},0,0) \text{ and} \\ \mu_7 = (2,-2,\overline{a_3},0,a_3,0,0) \end{cases}$$

where $0 \neq a_3 \in i\mathbb{R}$ as $\mu_6 \neq \mu_7$. Now a_3 must be a sum of two 8th roots of unity, so there are four posiblities: $a_3 = \pm 2i, \pm\sqrt{2}i$. Without loss of generality, there are really only the two possibilities $a_3 = 2i, \sqrt{2}i$ as we may switch μ_6 and μ_7. Now since $\ker(\mu_6) = \{1\}$, μ_6 is faithful, so $Z(\mu_6) = Z(G)$. Thus $a_3 \neq 2i$, so $a_3 = \sqrt{2}i$.

(c) Let $K = \langle r \rangle \cong \mathbb{Z}/8\mathbb{Z} \leq G$. We write the charater table of K, and we write the restricted characters of G below:

	1	r	r^2	r^3	r^4	r^5	r^6	r^7
$\lvert k \rvert$	1	1	1	1	1	1	1	1
ρ_1	1	1	1	1	1	1	1	1
ρ_2	1	ζ	i	ζ^3	-1	ζ^5	$-i$	ζ^7
ρ_3	1	i	-1	$-i$	1	i	-1	$-i$
ρ_4	1	ζ^3	$-i$	ζ	-1	ζ^7	i	ζ^5
ρ_5	1	-1	1	-1	1	-1	1	-1
ρ_6	1	ζ^5	i	ζ^7	-1	ζ	$-i$	ζ^3
ρ_7	1	$-i$	-1	i	1	$-i$	-1	i
ρ_8	1	ζ^7	$-i$	ζ^5	-1	ζ^3	i	ζ
$(\mu_1)\lvert_K$	1	1	1	1	1	1	1	1
$(\mu_2)\lvert_K$	1	1	1	1	1	1	1	1
$(\mu_3)\lvert_K$	1	-1	1	-1	1	-1	1	-1
$(\mu_4)\lvert_K$	1	-1	1	-1	1	-1	1	-1
$(\mu_5)\lvert_K$	2	0	-2	0	2	0	-2	0
$(\mu_6)\lvert_K$	2	$i\sqrt{2}$	0	$i\sqrt{2}$	-2	$-i\sqrt{2}$	0	$-i\sqrt{2}$
$(\mu_7)\lvert_K$	2	$-i\sqrt{2}$	0	$-i\sqrt{2}$	-2	$i\sqrt{2}$	0	$i\sqrt{2}$

where $\zeta = e^{\pi i/4}$. It is clear then that the $(\mu_i)|_K$'s are given by:

$$\begin{pmatrix} \mu_1 \\ \mu_2 \\ \mu_3 \\ \mu_4 \\ \mu_5 \\ \mu_6 \\ \mu_7 \end{pmatrix}_K = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} \rho_1 \\ \rho_2 \\ \rho_3 \\ \rho_4 \\ \rho_5 \\ \rho_6 \\ \rho_7 \\ \rho_8 \end{pmatrix}.$$

By the Frobenius Reciprocity Theorem, we have that

$$\begin{pmatrix} \rho_1 \\ \rho_2 \\ \rho_3 \\ \rho_4 \\ \rho_5 \\ \rho_6 \\ \rho_7 \\ \rho_8 \end{pmatrix}^G = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \mu_1 \\ \mu_2 \\ \mu_3 \\ \mu_4 \\ \mu_5 \\ \mu_6 \\ \mu_7 \end{pmatrix},$$

so $\mu_5 = \rho_7^G = \rho_3^G$, $\mu_6 = \rho_2^G = \rho_4^G$, and $\mu_7 = \rho_6^G = \rho_8^G$. Since all the other μ_i's are linear, we have that all representations are monomial, and G is an M-group.

Note: If we wanted to invoke Exercise XVIII.6, we would let N be the trivial group. $\square$