

MAT 252 - HW #19

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5. Let $1 \neq K \triangleleft G$ and $K \neq G$. Show that K is a Frobenius kernel (w.r.t. some Frobenius action of G) iff $C_G(x) \subset K \forall x \in K^* = K \setminus \{1\}$.

Proof:

\implies Suppose G acts on a finite set E Frobeniusly with K as the Frobenius kernel and let $\pi(\cdot)$ be the fixed point counter of this action.

Let $x \in K^*$. If $1 \neq g \in C_g(x)$, then $xg = gx$. Suppose $g \cdot a = a$ for some $a \in E$. Then,

$$x \cdot (g \cdot a) = x \cdot a \Rightarrow (xg) \cdot a = x \cdot a \Rightarrow g \cdot (x \cdot a) = x \cdot a$$

And since $\pi(x) = 0$, we see $x \cdot a \neq a$. So, $\pi(g) \geq 2$, contradicting G being Frobenius. Therefore, $g \cdot a \neq a$ for all $a \in E$. That is $\pi(g) = 0$ and so $g \in K$.

\Leftarrow We claim that $(|K|, [G : K]) = 1$. Suppose not, let a prime p divide both $|K|$ and $[G : K]$. By Cauchy's theorem, there exists an element $f \in K$ of order p . By Sylow's theorem, there exists a Sylow p -subgroup $P \subset G$ containing f . Now, if $x \in P \setminus K$, then $C_K(x) = 1$ because if $1 \neq k \in C_K(x)$, then $x = kxk^{-1} \Rightarrow xkx^{-1} = k \Rightarrow x \in C_G(k) \Rightarrow x \in K$, a contradiction. But then, $Z(P) \subset C_K(x) = 1$, contradicting that non-trivial p -group has non-trivial center. Thus, $P \subset K$ and $(|K|, [G : K]) = 1$

So, K is a hall subgroup. By Schur-Zassenhaus theorem, there exists H subgroup of G such that $K \rtimes H = G$. So, $H \cap K = \{1\}$.

Let $g \in G \setminus H$. If $x \in H \cap H^g$, then $x = g^{-1}hg$ for some $h \in H$. We can write $g = sk$ for some $s \in H$ and $k \in K^*$. Then, $x = k^{-1}s^{-1}hsk = k^{-1}tk$ where $t := s^{-1}hs \in H$. Since $K \triangleleft G$, we see $t^{-1}x = t^{-1}k^{-1}tk \in K$ while $t^{-1}x \in H$. So, $t^{-1}x = t^{-1}k^{-1}tk = 1 \Rightarrow t \in C_G(k) \in K$. Thus, $t = 1$ and $x = 1$. Hence, $H \cap H^g = \{1\}$ for all $g \in G \setminus H$.

So, H has the T.I. and G acts on G/H Frobeniusly by left multiplication by a theorem in class.

We check that K is the Frobenius kernel of this action.

Let $k \in K^*$. Suppose $kaH = aH$ for some $a \in G$ and some $f \in G \setminus K$. Then, $a^{-1}ka \in H$. Since $a^{-1}ka = b$ for some $b \in H$ and $k \in K \triangleleft G$, $b = a^{-1}ka \in K \cap H = \{1\} \Rightarrow k = 1$, a contradiction. Thus, $\pi(k) = 0$.

For $g \in G \setminus K$, we see that g must be in one of the conjugates of H because, by a counting argument, the number of elements outside of conjugates of H and K is

$$\left| G \setminus \left(\left(\bigcup_{x \in G} H^x \right) \cup K \right) \right| = |G| - (|K| + (|H| - 1)[G : H]) = |G| - |K| - |G| + [G : H] = 0.$$

(Recall that H is self-normalizing, so the number of conjugates of H is $[G : N_G(H)] = [G : H]$)

And so $g \in H^x$ for some $x \in G$, then $xgx^{-1} \in H \Rightarrow gx^{-1}H = x^{-1}H$, so $\pi(g) = 1$.