MAT 252 - HW #18

by Ka Choi

1. (For fun-loving folks!) A group G is called funny if it has n irreducible $\mathbb{C}$-characters with degree $1, 2, ..., n$. Show that all funny groups are trivial.

   Proof: Let G be funny. By Magic equation, we know $|G| = 1^2 + 2^2 + ... + n^2 = \frac{1}{6}n(n+1)(2n+1)$. By Master theorem, j divides $|G| = \frac{1}{6}n(n+1)(2n+1)$ for $j = 1, 2, ..., n$. In particular, $n \mid |G|$, so $\frac{1}{6}(n+1)(2n+1) \in \mathbb{N}$. Thus, n is odd otherwise $(n+1)(2n+1)$ is odd and is not divisible by 6. Let us assume $n \geq 3$ otherwise we are done. Since $n-1$ divides $|G|$, we have $\frac{n(n+1)(2n+1)}{6(n-1)} \in \mathbb{N} \Rightarrow \frac{n(n+1)(2n+1)}{n-1} \in \mathbb{N}$. Let $m = n-1$. So, $m \geq 2$ and is even. Then, $\frac{(m+1)(m+2)(2m+3)}{m} = \frac{2m^3+9m^2+13m+6}{m} \in \mathbb{N} \Rightarrow \frac{6}{m} \in \mathbb{N}$. Thus, $n-1 = m = 2, 6$. We only need to check for values of $n = 3, 7$:

   For $n = 3$, $|G| = 14$, not divisible by 3.

   For $n = 7$, $|G| = 140$, not divisible by 3.

   Hence, $n = 1$ and $|G| = 1$:-)