MAT 252 - HW #17

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7. Let $H \subset G$ and let V^* denote the contragradient kH-module of V. Show that $(V^*)^G \cong (V^G)^*$ as kG-module.

Proof: By Ex.10-5, if $\{e_1, ..., e_n\}$ is a basis for V and D is the matrix representation afforded by V, then w.r.t the dual basis $\{e_1^*, ..., e_n^*\}$, the contragradient representation is $D^*(g) = D(g^{-1})^t$. And we will denote $\dot{D}^*$ as the 0-extension of D^*. Thus, $\dot{D}^*(g) = \dot{D}(g^{-1})^t$.

In class, we have seen that $D^G(g) = (\dot{D}(g_i^{-1} g g_j))$ where g_i's are the coset representatives. Then,

$$(D^*)^G(g) = (\dot{D}^*(g_i^{-1} g g_j)) = (\dot{D}(g_j^{-1} g^{-1} g_i)^t)$$

while

$$(D^G)^*(g) = D^G(g^{-1})^t = (\dot{D}(g_i^{-1} g^{-1} g_j))^t = (\dot{D}(g_j^{-1} g^{-1} g_i)^t).$$

(Note that for a square matrix (A_{ij}) where A_{ij} is a block square matrix, its transpose $(A_{ij})^t = (A_{ji}^t)$.)

Therefore, $(D^*)^G(g) = (D^G)^*(g)$ which implies $(V^*)^G \cong (V^G)^*$.