MAT 252 - HW #17

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5. Let $H \subset G$. If $D : H \to O_n(k) = \{M \in \mathbb{M}_n(k) \mid M^t M = I_n\}$ is an orthogonal representation, show that D^G is also an orthogonal representation. If $D : H \to U_n(\mathbb{C})$ is a unitary representation, show that D^G is also a unitary representation.

   Proof: Let $m = [G : H]$. In class, we have seen that $D^G(g) = (\dot{D}(g_i^{-1} g g_j))$ where g_i's are the coset representatives. By assumption, $D(g)^t D(g) = I_n$. Fix a g. If $g \in H$, then $\dot{D}(g)^t \dot{D}(g) = D(g)^t D(g) = I_n$. If $g \notin H$, then $\dot{D}(g)^t \dot{D}(g) = 0$. Note also that $D^G(g)^t = (\dot{D}(g_i^{-1} g g_j))^t = (\dot{D}(g_j^{-1} g g_i)^t)$. And for each i, we claim that there is only one k such that $g_k^{-1} g g_i \in H$. Suppose not, that is, without loss of generality, $g_1^{-1} g g_i$ and $g_2^{-1} g g_i \in H$. Then, $g g_i \in g_1 H$ and $g g_i \in g_2 H$, a contradiction because $g_1 H \cap g_2 H = \emptyset$. Therefore, we have

$$(D^G(g)^t \; D^G(g))_{ii} = \sum_{k=1}^{m} \dot{D}(g_k^{-1} g g_i)^t \dot{D}(g_k^{-1} g g_i) = I_n$$

   And for $i \neq j$, we claim that only one of $g_k^{-1} g g_i$ and $g_k^{-1} g g_j$ can be in H. Suppose not, then $g_i, g_j \in g^{-1} g_k H$ which implies $i = j$ as we have chosen g_i, g_j to be the coset representatives, a contradiction. Thus,

$$(D^G(g)^t \; D^G(g))_{ij} = \sum_{k=1}^{m} \dot{D}(g_k^{-1} g g_i)^t \dot{D}(g_k^{-1} g g_j) = 0.$$

   Hence, $D^G(g)^t \; D^G(g) = I_{mn}$ and so $D^G(g)$ is orthogonal.

   The above argument goes through the same way if we replace transpose by conjugate-tranpose in the case of unitary representation.