MAT252 Lecture

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1 A Construction Idea

Let k be a field, D a finite-dimensional k-division algebra. Let G be a subgroup of D^* such that G k-spans D.

Then D is a simple kG-module.

Proof (trivial). If $0 \neq V \subset D$ is a kG-submodule, then V is closed under left kG-action. Therefore V is closed under left D-action, so V is a left ideal in D, so $V = D$. ☐

Example. *Let $G = \langle \sigma \rangle$, $o(\sigma) = n$. Construct all simple $\mathbb{Q}G$-modules.*

Solution. *Take $d \mid n$. Take $V_d = \mathbb{Q}(\zeta_d)$, where ζ_d is a primitive dth root of 1. Map $G \twoheadrightarrow \langle \zeta_d \rangle$ by $\sigma \mapsto \zeta_d$. $\langle \zeta_d \rangle$ $\mathbb{Q}$-spans V_d. Then V_d becomes a simple $\mathbb{Q}G$-module.*

$\dim_{\mathbb{Q}} V_d = \dim_{\mathbb{Q}} \mathbb{Q}(\zeta_d) = \varphi(d)$.

Set $\boxed{R = \mathbb{Q}G}$. Composition factors of ${}_R R$ include $\{V_d\}_{d|n}$. $\sum_{d|n} \dim V_d = \sum_{d|n} \varphi(d) = n = \dim_{\mathbb{Q}} R$. Conclusion: $\{V_d\}_{d|n}$ exhaust, without duplication, all simple $\mathbb{Q}G$-modules. (Note that the V_ds give inequivalent $\mathbb{Q}$-representations.)

Example. *Let G be the quaternion group $G = \{\pm 1, \pm i, \pm j, \pm k\}$. Construct all simple $\mathbb{R}G$-modules.*

At dimension 1, there are four $\mathbb{R}G$-representations, corresponding to the first four lines of the character table from a previous lecture.

Now, take $V = {}_{\mathbb{R}}\mathbb{H}$ where $\mathbb{H}$ = real quaternions. Then V is a simple 4-dimensional $\mathbb{R}G$-module. For $R = \mathbb{R}G$, the composition factors of ${}_R R$ include $\{V_1, V_2, V_3, V_4, V, \dots\}$. The sum of dimensions is $1 + 1 + 1 + 1 + 4 + \cdots$, so $\cdots$ is empty: there are exactly 5 simple $\mathbb{R}G$-modules.

Do a scalar extension from $\mathbb{R}$ to $\mathbb{C}$. V_1, V_2, V_3 and V_4 become their complex-homomorphisms. The last one becomes $\mathbb{C} \otimes_{\mathbb{R}} V = \mathbb{C} \otimes_{\mathbb{R}} \mathbb{H}$. Recall from last time that $\mathbb{C} \otimes_{\mathbb{R}} \mathbb{H} = \mathbb{M}_2(\mathbb{C})$.

$\left\{ \begin{pmatrix} * & * \\ * & * \end{pmatrix} \right\} = \left\{ \begin{pmatrix} * & 0 \\ * & 0 \end{pmatrix} \right\} \oplus \left\{ \begin{pmatrix} 0 & * \\ 0 & * \end{pmatrix} \right\}$ = *(left ideal)*$\oplus$*(right ideal).* $\left\{ \begin{pmatrix} * & * \\ * & * \end{pmatrix} \right\} = \left\{ \begin{pmatrix} * & 0 \\ * & 0 \end{pmatrix} \right\} \cong \left\{ \begin{pmatrix} 0 & * \\ 0 & * \end{pmatrix} \right\} \cong_{\mathbb{C}G\text{-module}} V_5 = 2$*-dim'l simple $\mathbb{C}G$-module.*

All previous discussion remains valid if $\mathbb{R}$ becomes $\mathbb{Q}$ and $\mathbb{C}$ becomes $\mathbb{Q}(\sqrt{-1})$.

Inside $\mathbb{H}^*$ there are other, bigger, finite subgroups. Most of them $\mathbb{R}$-span $\mathbb{H}$.

Example (Examples of such finite subgroups)**.**

- The generalized quaternion group of order $4m$. *(*$m = 2 \leftrightarrow$ *classical quaternion group.)* $1, i \in \mathbb{C} \subset \mathbb{H} \ni j$. *Conjugation by* j *induces complex conjugation on* $\mathbb{C}$. *Take inside* $\mathbb{C}^*$ *a cyclic subgroup of order* $2m = \langle \zeta_{2m} \rangle$. *Get* $\langle \zeta_{2m}, j \rangle$ *with order* $2m \cdot 2 = 4m$. *We get an irreducible* 2*-dimensional* $\mathbb{C}$*-representation (*4*-dimensional* $\mathbb{R}$*-representation) of this* G.
- *Binary tetrahedral, cardinality* 24. $\twoheadrightarrow A_4$. Binary *means it's a two-fold extension:* $1 \to \{\pm 1\} \to G \twoheadrightarrow A_4 \to 1$ *is an exact sequence of not-always-Abelian groups.*
- *Binary octahedral, cardinality* 48. $\twoheadrightarrow S_4$
- *Binary icosahedral, order* 120. $\twoheadrightarrow A_5$

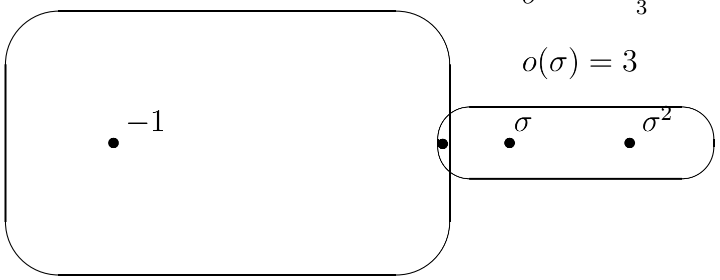
2 Constructing the Binary Tetrahedral Group

Hurwitz defined the "ring of integral quaternions" inside $R = \mathbb{Q} \cdot 1 \oplus \mathbb{Q} \cdot i \oplus \mathbb{Q} \cdot j \oplus \mathbb{Q} \cdot k$. Subrings of R contain a $\mathbb{Q}$-basis; are finitely generated as abelian groups. Take the maximal such object.

Hurwitz invented the following maximal order inside the $\mathbb{Q}$-quaternions. [FG page 5]. $A = \left\{ \frac{a \cdot 1 + b \cdot i + c \cdot j + d \cdot k}{2} \mid a, b, c, d \in \mathbb{Z} \text{ with the same parity} \right\}$. A turns out to be maximal (have maximal order).

This has $\mathbb{Z}$-basis $\left\{ i, j, k, \frac{1+i+j+k}{2} \right\}$.

Units of A: $U(A) = \{\pm 1, \pm i, \pm j, \pm k, \frac{\pm 1 \pm i \pm j + \pm k}{2} \text{ with arbitrary signs}\}$. $|U(A)| = 8 + 2^4 = 24$. This is the binary tetrahedral group.


$\sigma = \frac{-1+i+j+k}{3}$

$o(\sigma) = 3$

$Q_8 \triangleleft$ binary tetrahedral group