Math 252 - Exercises XIX
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(**4**) Let G be a p-group acting transitively on a non-singleton set. Show that the Frobenius kernel of the action has cardinality $\geq 1+\varphi(|G|)$, where φ is Euler's totient function.

Solution: Because the action is transitive, we may assume that the G-set is the left coset space G/H for some subgroup $H \subsetneq G$. As usual we let K denote the Frobenius kernel of the action. Because H is proper, a standard result from Sylow theory (essentially, the fact that p-groups are solvable) implies that there exists $N \trianglelefteq G$ with $[G:N]=p$ such that $H \subseteq N$. We claim that $G \setminus N \subseteq K \setminus \{1\}$. Indeed, let $g \in G \setminus N$, and assume for contradiction that $g \cdot xH = xH$ for some $xH \in G/H$. Then $g^x \in H \subseteq N$, and because N is normal we have $g = (g^x)^{x^{-1}} \in N$, a contradiction.

Let $|G| = p^n$, so that $|N| = p^{n-1}$. Then since $G \setminus N \subseteq K \setminus \{1\}$, we have

$$\begin{aligned}\varphi(p^n) &= p^n - p^{n-1}\\ &= |G| - |N|\\ &\leq |K| - 1,\end{aligned}$$

which proves the claim that $|K| \geq 1+\varphi(|G|)$.