

5. Let K be a proper normal subgroup of G . Show that K is a Frobenius kernel (w.r.t. some Frobenius action of G) iff $C_G(x) \subseteq K \forall x \in K \setminus \{1\}$.

Proof. The forward implication was proved in class. To prove the converse, consider the conjugation action of G on K . Since $C_G(x) \subseteq K$ for $x \neq 1$, all nontrivial orbits of this action have length divisible by $[G : K]$. Therefore $[G : K] \mid (|K| - 1)$ which implies $([G : K], |K|) = 1$. We can then use the Schur-Zassenhaus theorem to write $G = K \rtimes H$ for some subgroup H of G .

Consider the natural action of G on the left coset space $G/H = \{kH \mid k \in K\}$. Clearly K acts regularly on G/H , so it suffices to prove that H has the T.I. property (then G/H is a Frobenius G -set, and K is contained in the Frobenius kernel of the G -action on G/H ; but $|K| = |G/H|$ which shows that K is the Frobenius kernel of this action). Assume H^g intersects H nontrivially for some $g = hk$, $h \in H$, $k \in K$. Since $H^g = H^k$, we can find some $1 \neq t \in H$ s.t. $k^{-1}tk \in H$. Then $t^{-1}k^{-1}tk \in H$, and since $H \cap K = \{1\}$ and $t^{-1}k^{-1}t \in K$, we get $t^{-1}k^{-1}tk = 1$, hence $t \in C_G(k) \subseteq K$. But this is clearly impossible since $t \in H$ and $t \neq 1$. □