2. Let $H \subseteq G$ be groups and let k be any field (with no assumptions on char(k)). Let $d_k(G)$ be the largest dimension of a simple kG-module. Show that $d_k(G) \leq [G : H]d_k(H)$. (For instance, if k is algebraically closed and H is abelian, then $d_k(G) \leq [G : H]$.)

Proof. Let $0 = M_0 \subset M_1 \subset ... \subset M_r = kH$ be a composition series for kH and for each $i = \overline{0, r}$ let d_i be the dimension of M_i over k. Since $(kH)^G = kG$, inducing the M_i's up to G we get an increasing sequence of G-modules $0 = M_0^G \subset ... \subset M_r^G = kG$ which can be refined to a composition series of kG. We have $dim_k(M_i^G) = [G : H]d_i$, hence

$$\begin{aligned} d_k(G) &\leq \max_{i=\overline{0,r-1}} (dim_k(M_{i+1}^G) - dim_k(M_i^G)) \\ &= [G : H] \max_{i=\overline{0,r-1}} (d_{i+1} - d_i) \\ &= [G : H]d_k(H). \end{aligned}$$

□