8. Let $H \lhd G$ and $\mu \in Irr(G)$. Show that $\mu = \nu^G$ for some $\nu \in Irr(H)$ iff $\mu(G \setminus H) = \{0\}$ and μ_H is a sum of distinct characters in $Irr(H)$.

Proof. Assume first that $\mu = \nu^G$ for some $\nu \in Irr(H)$. We have seen in class that since H is a normal subgroup of G we must have $\mu(G \setminus H) = \{0\}$. Let V be an H-module affording the representation whose character is μ. Since H is normal we have $HsH = sHH = sH \ \forall s \in G$, thus the double-coset decomposition is just

$$G = \bigcup_{s \in S} sH,$$

where S is a complete set of representatives of G/H. Using the equalities $H_s = H \ \forall s \in S$ we can easily see that the decomposition in Mackey's Subgroup Theorem is

$$(V^G)_H = \bigoplus_{s \in S} sV.$$

The H-modules V and sV have no common composition factor for $s \neq 1$ by Mackey's Irreducibility Criterion. Let ν_s denote the character of the representation afforded by the H-module sV, $s \in S$. Then $\nu_s(h) = \nu(h^s)$ and an easy computation shows that $[\nu_s, \nu_t] = [\nu, \nu_{s^{-1}t}] = 1$ if $s = t$ and 0 otherwise. It follows that the ν_s's are irreducible and distinct, and

$$\mu_H = \sum_{s \in S} \nu_s.$$

For the converse, assume that $\mu(G \setminus H) = \{0\}$ and $\mu_H = \sum_{i=1}^{r} \nu_i$ with ν_i distinct irreducible characters of H. If U is the G-module affording the representation corresponding to μ, we have the decomposition

$$U = \bigoplus_{i=1}^{r} V_i$$

with V_i simple H-module whose character is ν_i, $i = \overline{1,r}$. If we can show that G permutes the V_i's transitively and that the isotropy subgroup of V_1 is H, then the conclusion follows from the Recognition Criterion for an induced module.

Let $v \in V_i$ and let $g \in G$. Then $gv = w \in V_j$ for some j. We have

$$gV_i = g(\mathbb{C}Hv_i) = \mathbb{C}H(gv_i) = \mathbb{C}Hw = V_j$$

since V_i, V_j are simple and H is normal. If the action weren't transitive, taking the sum of the modules in an orbit would give us a nontrivial G-submodule of U, contradicting its simplicity.

Now assume there is some $g \notin H$ s.t. $gV_1 = V_1$. Let $\phi \in \mathbb{C}H$ which acts as g^{-1} on V_1 and as 0 on the other V_i's. Since $\mu(G \setminus H) = \{0\}$ we get that $Tr(g\phi) = 0$ (here we see $g\phi$ as an endomorphism of U), but $g\phi$ is the identity on V_1 and 0 on the other V_i's, therefore $Tr(g\phi) = dim\ V_1 \neq 0$. This contradiction concludes the proof.

□