8. Use Exercise 6 to prove the following extension of a result obtained in class discussions: Suppose a group G has an abelian p-Sylow group P (for some prime p). If there exists $\chi \in Irr(G)$ with $\chi(1) = p^r$ $(r \geq 1)$, then the p-part of $[G : Z(\chi)]$ is p^r.

Proof. By the generalized Schur's theorem, $p^r = \chi(1) | [G : Z(\chi)]$, so it suffices to prove that the p-part of $[G : Z(\chi)]$ is at most p^r.

Let Q be a p-Sylow subgroup of $Z(\chi)$. Replacing eventually P by some conjugate we may assume that Q is contained in P (here we used Sylow's theorems). The p-part of $[G : Z(\chi)]$ is then $[P : Q]$ and $P \cap Z(\chi) = Q$. If we restrict χ to P we get the character of some p^r-dimensional representation of P. Let $g \in P \setminus Q$. Then $g \notin Z(\chi)$, and since g is prime to χ (the centralizer of g contains P since P is abelian, hence $(|g|, p) = 1$) we get $\chi(g) = 0$. Now we can use Exercise 6 to get $[P : Q] | p^r$ and the conclusion follows.

□