3. Let $U, V, W$ be simple $kG$-modules, where $k$ is a splitting field of characteristic 0 for $G$. Let $T$ be the regular module $kG$.

(a) Show that $U \otimes T \simeq (dim\ U) \cdot T$.

(b) Show that $U \otimes V$ can be embedded in $T$.

(c) Show that $W$ appears in $U \otimes V$ at most $dim\ W$ times.

*Proof.* (a) Since $char(k) = 0$ it suffices to show that $\chi_{U \otimes T} = \chi_{(dim\ U) \cdot T}$, which is clear ($\chi_T(g) = 0$ for $g \neq 1$ and $\chi_U(1) = dim\ U$, then $\chi_{U \otimes T}(g) = \chi_U(g)\chi_T(g) = 0 = (dim\ U)\chi_T(g)$ for $g \neq 1$, and $\chi_{U \otimes T}(1) = \chi_U(1)\chi_T(1) = (dim\ U)\chi_T(1)$).

(b) follows from (c) since every simple $kG$-module $W$ appears in $T$ exactly $dim\ W$ times.

(c) If $dim\ W$ is greater than or equal to both $dim\ U$ and $dim\ V$, then $W$ appears in $U \otimes V$ at most $dim\ U \otimes V / dim\ W = dim\ U \cdot dim\ V / dim\ W \leq dim\ W$. Assume otherwise that $dim\ U$ is the greatest of the three dimensions. Then $W$ appears in $U \otimes V$ exactly $[\chi_{U \otimes V}, \chi_W] = [\chi_U \chi_V, \chi_W] = [\chi_U, \chi_{\overline{V}} \chi_W]$ which is the number of times $U$ appears in $\overline{V} \otimes W$. This is at most $dim\ \overline{V} \cdot dim\ W / dim\ U$ which is at most $dim\ W$ since $dim\ \overline{V} = dim\ V \leq dim\ U$.

□