Math 252 Assignment 20

12/10/07 David Penneys

(3) Let $G = K \rtimes H$ wher $H = \langle h \rangle$ has order 2, K is an abelian group of odd order, and $hxh^{-1} = x^{-1}$ for all $x \in K$. Show that G is Frobenius with kernel K and complement H.

Note: Taking $K = \mathbb{Z}/3\mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z}$ gives the "other" Frobenius group of order 18 discussed in class.

Solution. We write elements in G as (k,h) where $k \in K,\ h \in H$. We then idenitfy $K = \big\{(k,1)\big|k \in K\big\}$ and $H = \big\{(1,f)\big|f \in H\big\}$ by a slight abuse of notation. It suffices to show:

(a) $C_G((x,1)) \subset K$ for all $(x,1) \in K \setminus \{(1,1)\}$, and

(b) H is a T.I. subgroup, i.e. $H \cap H^{(k,f)} = \{(1,1)\}$ for all $(k,f) \notin H$.

(a): Let $(x,1) \in K$. Then $(k,h) \in C_G((x,1))$ if and only if

$$(x,1) = (k,h)(x,1)(k,h)^{-1} = (khxh^{-1}hkh^{-1},1) = (kx^{-1}k^{-1},1) = (x^{-1},1)$$

if and only if $x = 1$ since $|K|$ is odd.

(b): Suppose $(k,f) \notin H$ (if and only if $(k,f)^{-1} \notin H$). Then $H \cap H^{(k,f)^{-1}} = \{(1,1)\}$ if and only if

$$(1,h) \neq (k,f)(1,h)(k,f)^{-1}.$$

Case 1: Suppose $f = 1$. Then

$$(1,h) = (k,1)(1,h)(k,1)^{-1} = (k,h)(k^{-1},1) = (k^2,h)$$

if and only if $k = 1$, which cannot happen as $(k,f) \notin H$.

Case 2: Suppose $f = h$. Then

$$(1,h) = (k,h)(1,h)(k,h)^{-1} = (k,1)(k,h) = (k^2,h)$$

if and only if $k = 1$, which cannot happen as $(k,f) \notin H$. $\square$