Math 252 Assignment 16

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(1) For any $\chi \in \mathrm{Irr}(G)$, show that

$$d = \prod_{\neq 1} \chi(g) \in \mathbb{Z}.$$

If $\chi(1) = 1$, show that $d = \pm 1$, and that both values are possible.

Solution. Lin, Lee, and Yoo's solution is outstanding. We provide a proof of the second part not assuming the first part as the technique is fun and interesting (although a little roundabout and tedious). Therefore, we show:

(1') For $\chi \in \mathrm{Irr}(G)$ with $\chi(1) = 1$, show that

$$d = \prod_{g \neq 1} \chi(g) = \pm 1.$$

Proof of (1'). Since $\chi(1) = 1$, it comes from a one dimensional representation which factors uniquely through the canonical epimorphism $\varphi\colon G \to [G, G]$, i.e., there is a representation D of G/H where $H = [G, G]$ such that $\chi(g) = \chi_D(gH)$ for all $g \in G$. Since $|H|$ elements of G map to the same element in G/H, we have that

$$d = \prod_{g \neq 1} \chi(g) = \prod_{g \neq 1} \chi_D(gH) = \prod_{g \notin H} \chi_D(gH) = \left(\prod_{gH \neq H} \chi_D(gH) \right)^{|H|}$$

as $\chi(g) = 1$ for all $g \in H$. Hence, it suffices to consider the case where G is abelian, since if the result holds for abelian groups, then $d = (\pm 1)^{|H|} = \pm 1$ as G/H is abelian.

Suppose G is abelian. Then by the structure theorem for finitely generated abelian groups, we have

$$G \cong \prod_{j=1}^{n} \mathbb{Z}/n_j\mathbb{Z}$$

as G is finite. Identifying G with the direct product of cyclic groups, G is the disjoint union

$$G = \{1\} \amalg \coprod_{j=1}^{n} \big((\mathbb{Z}/n_j\mathbb{Z}) \setminus \{1\} \big).$$

Hence, we may reduce to the case where G is cyclic, since if $K_j = (\mathbb{Z}/n_j\mathbb{Z}) \setminus \{1\}$,

$$d = \prod_{g \neq 1} \chi(g) = \prod_{j=1}^{n} \prod_{g \in K_j} \chi(g),$$

and if the result is true for a cyclic group, then d is the product of j integers each of which are ± 1, and thus $d = \pm 1$.

Suppose $G = \mathbb{Z}/n\mathbb{Z}$. Then if g is a generator of G, then $\chi(g) = \zeta^k$ for some $k = 0, 1, \ldots, n-1$ where $\zeta = e^{2\pi i/n}$. Now

$$\begin{aligned} d = \prod_{g \neq 1} \chi(g) = \prod_{j=1}^{n-1} \chi(g^j) = \prod_{j=1}^{n-1} \chi(g)^j = \prod_{j=1}^{n-1} \zeta^{jk} = \prod_{j=1}^{n-1} e^{2\pi i k/n} \\ = \exp\left(\frac{2\pi i}{n} \sum_{j=1}^{n-1} j \right) = \exp\left(\frac{2\pi i}{n} \frac{(n-1)n}{2} \right) = e^{(n-1)\pi i} = \pm 1 \end{aligned}$$

as desired. $\square$