Math 252: Exercises XIX by Dmitry Pavlov, pavlov@math.

Problem 6. Let G be a Frobenius group with kernel K and complement H. Show that $Z(G)$ is trivial and for any y in $G \setminus K$ the conjugation action of y on K defines an outer automorphism of K.

Solution. From exercise 5 it follows that $Z(G) \subset \cap_{x \in K \setminus \{1\}} C_G(x) \subset K$. If $k \in K$ belongs to $Z(G)$, then it commutes with all elements of H. But this is impossible when $k \notin H$, because then we would have $H \cap H^k = \{1\}$ by trivial intersection property, contradicting the fact that k commutes with the elements of H. Hence $k \in H$ and because $K \cap H = \{1\}$ we have $k = 1$. Therefore $Z(G) = \{1\}$.

Now suppose that the conjugation action of $y \in G \setminus K$ on K defines an inner automorphism of K. Then we have $yxy^{-1} = kxk^{-1}$ for some $k \in K \setminus \{1\}$ and for all $x \in K$. In particular, if $x = k$, then $yky^{-1} = k$ and $yk = ky$. Because $C_G(k) \subset K$ for all $k \in K \setminus \{1\}$ we have $y \in K$. Contradiction.