Math 252: Exercises XVII by Dmitry Pavlov, pavlov@math.

Problem 8. If G is a finite group, $H = G \times G$, $U = kG$ is a kH-module via the action $(g, h)a = gah^{-1}$, then U is a monomial module over kH.

Solution. Let P be a subgroup of H formed by the elements (g, g). We shall show now that U is a monomial module over kH with respect to the trivial representation Q of P. Indeed, define a map $f \colon kH \otimes_k Q \to U$ as follows: $(g, h) \otimes 1 \mapsto gh^{-1}$. Now note that $(g, h)(p, p) \otimes 1 = (gp, hp) \otimes 1 \mapsto gpp^{-1}h^{-1} = gh^{-1}$, hence the map factors through $kH \otimes_{kG} Q$. We now have a map from $kH \otimes_{kG} Q$ to U. Now note that the last map is a surjective homomorphism of modules of the same dimension over k, hence it is an isomorphism of vector spaces. At last, $(a, b)(g, h) \otimes 1 = (ag, bh) \otimes 1 \mapsto agh^{-1}b^{-1} = (a, b) \cdot gh^{-1}$, hence this isomorphism is an isomorphism of modules.