Math 252 Representation Theory: Exercises XVI Q1

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For any $\chi \in \mathrm{Irr}(G)$, show that $d := \prod_{g\neq 1} \chi(g) \in \mathbb{Z}$. If $\chi(1) = 1$, show that $d = \pm 1$, and that both values are possible.

Proof:

Let $\chi \in \mathrm{Irr}(G)$. Since $\chi(g)$ is an algebraic integer for all $g \in G$, the product $d := \prod_{g\neq 1} \chi(g)$ is also an algebraic integer. Thus, it suffices to show that $d \in \mathbb{Q}$.

Let $r = \chi(1)$, $n = |G|$ and $\zeta = e^{2\pi i/n}$. Each $\chi(g)$ is a sum of n-th roots of unity so

$$\chi(g) = \zeta^{n_1} + \ldots + \zeta^{n_r}$$

for some integers $n_1, \ldots, n_r$. Furthermore, given $\sigma \in \mathrm{Gal}(\mathbb{Q}(\zeta)/\mathbb{Q}) = (\mathbb{Z}/n\mathbb{Z})^\times$, we have $\sigma(\zeta) = \zeta^k$ for some k coprime to n. Hence,

$$\sigma\chi(g) = \zeta^{kn_1} + \ldots + \zeta^{kn_r} = \chi(g^k).$$

Now, as g ranges over $G\backslash\{1\}$, g^k also ranges over $G\backslash\{1\}$. Indeed, we can write $pk + qn = 1$ for some $p, q \in \mathbb{Z}$, so if $g_1^k = g_2^k$ for some $g_1, g_2 \in G$, then $g_1 = (g_1^k)^p(g_1^n)^q = (g_2^k)^p(g_2^n)^q = g_2$. Thus, for each $\sigma \in \mathrm{Gal}(\mathbb{Q}(\zeta)/\mathbb{Q})$,

$$\sigma d = \sigma \prod_{g\neq 1} \chi(g) = \prod_{g\neq 1} \chi(g^k) = \prod_{g\neq 1} \chi(g) = d$$

with k coprime to n. This proves that d is in the fixed field $\mathbb{Q}$ of $\mathrm{Gal}(\mathbb{Q}(\zeta)/\mathbb{Q})$, as required.

If $\chi(1) = 1$, each $\chi(g)$ is an n-th root of unity. Thus, $|d| = \prod_{g\neq 1} |\chi(g)| = 1$. Since d is integer, we have $d = \pm 1$. The character table of $\mathbb{Z}/2\mathbb{Z}$ shows that both values are possible:

	1	g
χ_1	1	1
χ_2	1	-1