Math 252: Representation Theory

Exercises XX

Problem 4. *If (G, K, H) is a "Frobenius Triple," show that the Frobenius kernel K and the Frobenius G-set G/H are both uniquely determined.*

Definition (p-Core)**.** Let p be a prime. The p-core of a finite group G is the largest normal p-subgroup of G and is denoted $\mathbf{O}_p(G)$. If P is a Sylow p-subgroup of G, we may calculate p-core of G as

$$\mathbf{O}_p(G) = \bigcap_{g \in G} P^g.$$

Observe $\mathbf{O}_p(G) \lhd G$ by construction.

Definition (Fitting Subgroup)**.** Let G be a finite group. We define the Fitting subgroup $\mathbf{F}(G)$ as the product of all p-cores of G, i.e.

$$\mathbf{F}(G) = \prod_{p \mid |G|} \mathbf{O}_p(G).$$

Since each $\mathbf{O}_p(G)$ is normal in G, this product is well-defined. Additionally, $\mathbf{F}(G) \lhd G$.

Lemma 1. *Let G be a finite group. Then $\mathbf{F}(G)$ is the unique largest normal nilpotent subgroup of G. That is, if N is a nilpotent normal subgroup of G then $N \subseteq \mathbf{F}(G)$.*

Proof. It suffices to verify that each Sylow p-subgroup of N is contained in $\mathbf{F}(G)$. Let P be a Sylow p-subgroup of N. Since N is nilpotent $P \lhd N$. In fact P is a characteristic subgroup of N and hence is normal in G. Thus $P \subseteq \mathbf{O}_p(G) \subseteq \mathbf{F}(G)$. □

Lemma 2. *Let (G, K, H) be a "Frobenius Triple." Then $\mathbf{O}_p(G) \subseteq K$ for all p and in particular $\mathbf{F}(G) \subseteq K$.*

Proof. Let p be any prime. Suppose $p \nmid |K|$ and let P be a Sylow p-subgroup of H. Then P is a Sylow p-subgroup of G and

$$\mathbf{O}_p(G) = \bigcap_{g \in G} P^g \subseteq \bigcap_{g \in G} H^g = \{1\} \subset K.$$

If instead $p \mid |K|$ we let P be a Sylow p-subgroup of K. Then P is a Sylow p-subgroup of G and

$$\mathbf{O}_p(G) = \bigcap_{g \in G} P^g \subseteq \bigcap_{g \in G} K^g = K.$$ □

Solution by Bradley Froehle (bfroehle@math.berkeley.edu), with some revisions suggested by Prof. Lam.

Lemma 3. *Let (G, K, H) be a "Frobenius Triple." Then $K = \mathbf{F}(G)$ and hence K is uniquely determined from G.*

Proof. A theorem of J.G. Thompson gives that K is a nilpotent normal subgroup of G and hence $K \subseteq \mathbf{F}(G)$ by Lem. 1. The reverse inclusion is established by Lem. 2. □

Lemma 4. *Let (G, K, H) be a "Frobenius Triple." The Frobenius G-set G/H is uniquely determined.*

Proof. The details are left to the reader.[1] □

[1] Actually, I don't know how to show this at the moment and I don't have my class notes with me.