Math 252: Representation Theory

Exercises XIX

Problem 3. *If χ is a monomial character of G over any field k, show that χ vanishes on at least $\chi(1) - 1$ elements of G.*

Proof. Suppose $\chi = \mu^G$ where μ is a linear character of $H \subseteq G$. We'll prove that χ vanishes on at least $[G : H] - 1$ elements of G, and hence χ vanishes on at least $\chi(1) - 1$ elements of G.

Let $g_1, \ldots, g_r$ be left coset representatives of H in G, so that $G = \dot{\bigcup} g_i H$. Recall that the induced character given by

$$\chi(g) = \sum_{i=1}^{r} \dot{\mu}(g^{g_i}).$$

Observe that if $g \in G \setminus \left(\bigcup_{t \in G} H^t\right)$, then $g^{g_i} \notin H$ for any i so in particular $\chi(g) = 0$. Jordan's theorem tells us that that $|G \setminus \bigcup H^t| \geq [G : H] - 1$, and therefore χ vanishes on at least $[G : H] - 1$ elements of G.

We complete the proof by noting that $\chi(1) = \sum_{i=1}^{r} \dot{\mu}(1) \leq [G : H]$, and thus

$$\chi(1) - 1 \leq [G : H] - 1 \leq \left|\{g \in G \mid \chi(g) = 0\}\right|$$

as desired. □

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