Math 252: Representation Theory

Exercises XVIII

Problem 6. *For $N \subseteq H \subseteq G$ with $N \triangleleft G$, let $\bar{G} = G/N$ and $\bar{H} = H/N$. For any $k\bar{H}$-module $\bar{V}$, let V denote $\bar{V}$ viewed as a kH-module by pullback. Show that V^G is kG-isomorphic to the pullback of $\bar{V}^{\bar{G}}$.*

Proof. Consider the homomorphism $\varphi : kG \otimes_{kH} V \to k\bar{G} \otimes_{k\bar{H}} \bar{V}$ given by $\varphi(g \otimes v) := \bar{g} \otimes v$ and extended by linearity. We verify several facts about φ:

- **Well-defined:** We need to verify middle linearity, i.e. if $h \in H$ then φ maps $gh \otimes v$ and $g \otimes hv$ to the same element. Consider:

$$\varphi(gh \otimes v) = \overline{gh} \otimes v = \bar{g}\bar{h} \otimes v = \bar{g} \otimes \bar{h}v = \bar{g} \otimes hv = \varphi(g \otimes hv).$$

- ***kG*-homomorphism:** Let $x \in G$. We verify $x \cdot \varphi(g \otimes v) = \varphi(x \cdot g \otimes v)$, as

$$x \cdot \varphi(g \otimes v) = x \cdot (\bar{g} \otimes v) = \bar{x}\bar{g} \otimes v = \overline{xg} \otimes v = \varphi(x \cdot g \otimes v).$$

- **Surjective:** Let $\bar{g} \in \bar{G}$ and $v \in V$. Then if $g \in G$ is any element projecting to $\bar{g}$ under the canonical surjection, $\varphi(g \otimes v) = \bar{g} \otimes v$. Since elements of this type form a k-basis for $\bar{V}^{\bar{G}}$, we conclude that φ is surjective.

- **Dimensions agree:** Rather than show that φ is injective, we show that the dimensions (over k) of the domain and range are equal. Observe $\dim_k V^G = [G : H] \dim_k V = [\bar{G} : \bar{H}] \dim_k \bar{V} = \dim_k \bar{V}^{\bar{G}}$.

Thus φ is a kG-isomorphism and V^G and $\bar{V}^{\bar{G}}$ are kG-isomorphic.

□

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