Math 252: Representation Theory

Exercises XVIII

Problem 2. *For groups $H \subseteq G$, let $\nu \in F_k(H)$, where* $\operatorname{char} k \nmid |G|$. *For $g \in G$, show that $\nu^G(g) = |H|^{-1}|C_G(g)| \sum_{h \in H \cap C} \nu(h)$, where C denotes the conjugacy class of g in G.*

Proof. Recall the usual formula for the induced character in the case $\operatorname{char} k \nmid |G|$:

$$\nu^G(g) = \frac{1}{|H|} \sum_{t \in G} \dot{\nu}(g^t).$$

Let C be the conjugacy class of g in G. Clearly $g^t \in C$. Also note that $\dot{\nu}(g^t)$ is zero unless $g^t \in H$. Thus we may rewrite the formula for $\nu^G(g)$ as

$$\nu^G(g) = \frac{1}{|H|} \sum_{\substack{t \in G \\ g^t \in H \cap C}} \nu(g^t).$$

Ideally we would like to switch this summation to be of the form $\sum_{h \in H \cap C} \nu(h)$, however several $t \in G$ might satisfy $g^t = h$. We may count the number of duplicates by the orbit-stabilizer theorem: There are exactly $|C_G(g)|$ such t which satisfy $g^t = h$ for some fixed $h \in C$.

Therefore

$$\nu^G(g) = \frac{1}{|H|} \sum_{\substack{t \in G \\ g^t \in H \cap C}} \nu(g^t) = \frac{1}{|H|} |C_G(g)| \sum_{h \in H \cap C} \nu(h)$$

as desired. □

Solution by Bradley Froehle (`bfroehle@math.berkeley.edu`).