Math 252: Representation Theory

Exercises XVIII

Problem 1 (For fun-loving folks!)**.** *A group G is called* funny *if it has n irreducible $\mathbb{C}$-characters with degrees $1, 2, 3, \ldots, n$. Show that all funny groups are trivial.*

Proof. Suppose such a funny group G exists. Clearly the trivial group is funny, so we exclude that case and consider only the situation $n > 1$. We calculate the order of the group using the magic equation:

$$|G| = 1^2 + 2^2 + \cdots + n^2 = \frac{1}{6}n(n+1)(2n+1).$$

Since $\mathbb{C}$ is a splitting field for G, we know that the degree of each character must divide the order of the group. In particular $(n-1) \mid |G|$. However $|G| \mid n(n+1)(2n+1)$, and hence we must have $(n-1) \mid n(n+1)(2n+1)$. A quick calculation shows

$$n(n+1)(2n+1) = 2(n-1)^3 + 9(n-1)^2 + 13(n-1) + 6$$

and therefore $(n-1) \mid 6$.

This gives 4 possible cases for n. We examine each possibility:

- $n = 2$: We calculate $|G| = 5$, but $2 \nmid 5$ so no such group exists.
- $n = 3$: We calculate $|G| = 14$, but $3 \nmid 14$ so no such group exists.
- $n = 4$: We calculate $|G| = 30$, but $4 \nmid 30$ so no such group exists.
- $n = 7$: We calculate $|G| = 140$, but $3 \nmid 140$ so no such group exists.

We conclude that the only funny group is the trivial group[1]. □

Solution by Bradley Froehle (bfroehle@math.berkeley.edu). Thanks to Dave Penneys for noticing an error in a previous version of this document.

[1] I always knew there was something fishy going on with the trivial group....