

# Math 252: Representation Theory

## Exercises XVII

**Problem 8.** Let  $G$  be any finite group, and let  $\Gamma = G \times G$ . Let  $U = kG$  viewed as a  $k\Gamma$ -module via the action  $(g_1, g_2) \cdot \alpha = g_1 \alpha g_2^{-1}$ . Show that  $U$  is a monomial module over  $k\Gamma$ .

*Proof.* Let  $\Delta = \{(g, g) \mid g \in G\} \subseteq \Gamma$ . We claim that  $U$  is isomorphic to the trivial module of  $\Delta$  induced up to  $\Gamma$ , i.e.  $U \cong (1_\Delta)^\Gamma$ .

Recall that  $(1_\Delta)^\Gamma = k\Gamma \otimes_{k\Delta} k$  where we endow  $k$  with the trivial  $\Delta$  action. Define

$$\begin{aligned} \varphi : k\Gamma \otimes_{k\Delta} k &\longrightarrow U \\ (x, y) \otimes 1 &\mapsto xy^{-1} \end{aligned}$$

and extend by linearity.

We verify the following facts about  $\varphi$ :

- **Well-defined:** If  $(g, g) \in \Delta$ , then  $(x, y) \otimes 1 = (xg, yg) \otimes 1$  and both are mapped to  $xy^{-1}$  under  $\varphi$ .
- **Surjective:**  $\varphi((g, 1) \otimes 1) = g$  for any  $g \in G$ .
- **Injective:** If  $\varphi((a, b) \otimes 1) = \varphi((x, y) \otimes 1)$  (that is,  $ab^{-1} = xy^{-1}$ ), then  $(a, b) \otimes 1 = (ab^{-1}, 1) \otimes 1 = (xy^{-1}, 1) \otimes 1 = (x, y) \otimes 1$ .
- **$k\Gamma$ -module homomorphism:** Let  $(g_1, g_2) \in \Delta$ . Then

$$(g_1, g_2) \cdot \varphi((\alpha, \beta) \otimes 1) = (g_1, g_2) \cdot \alpha\beta^{-1} = g_1(\alpha\beta^{-1})g_2^{-1}$$

and

$$\varphi((g_1, g_2) \cdot (\alpha, \beta) \otimes 1) = \varphi((g_1\alpha, g_2\beta) \otimes 1) = g_1(\alpha\beta^{-1})g_2^{-1}.$$

Thus  $\varphi$  is a  $k\Gamma$ -module isomorphism, and hence  $U$  is a monomial module over  $k\Gamma$ .  $\square$

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