Math 252: Representation Theory

Exercises XVII

Problem 7. *Let $H \subseteq G$ and let V^* denote the contragredient kH-module of V. Show that $(V^*)^G \cong (V^G)^*$ as kG-modules.*

Note. Since Ka Choi already provided a solution by explicitly computing the matricial representations for the two modules in question, here we present a different proof which holds in the case $\operatorname{char} k = 0$.

Proof. We prove the claim for the case $\operatorname{char} k = 0$.

Let χ_V the the character afforded by V. We verify that the characters afforded by $(V^G)^*$ and $(V^*)^G$ agree by explicitly computing their values.

$$\begin{aligned}
\chi_{(V^G)^*}(g) &= \chi_{V^G}(g^{-1}) \\
&= \frac{1}{|H|} \sum_{x \in G} \dot{\chi}_V(x^{-1} g^{-1} x) \\
&= \frac{1}{|H|} \sum_{x \in G} \dot{\chi}_{V^*}(x^{-1} g x) \\
&= \chi_{(V^*)^G}(g).
\end{aligned}$$

Since the two characters agree and we are working over a field of characteristic zero, we conclude that the two modules are kG-isomorphic. □

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