Math 252: Representation Theory

Exercises XVII

Problem 4. *Let $H \subseteq G$ and let V be a kH-module. Make $U := \mathrm{Hom}_{kH}(kG, V)$ into a left kG-module by $(g \cdot f)(\alpha) = f(\alpha g)$ ($g \in G$, $f \in U$, $\alpha \in kG$). Show that U is kG-isomorphic to the induced module $V^G = kG \otimes_{kH} V$.*

Proof. Let $x_1, x_2, \ldots, x_r$ be a complete set of coset representatives for H in G, so that $G = \bigcup x_i H = \bigcup H {x_i}^{-1}$. Recall that we may write $V^G = \oplus_{i=1}^r (x_i \otimes V)$ as k-supspaces.

We define a kG-module homomorphism $\varphi : \mathrm{Hom}_{kH}(kG, V) \to kG \otimes_{kH} V$ by

$$\varphi(f) = \sum_{i=1}^r x_i \otimes f({x_i}^{-1}).$$

Clearly φ is additive: $\varphi(f + f') = \varphi(f) + \varphi(f')$.

We verify that φ is a kG-module homomorphism, as if $g \in G$ then

$$\begin{aligned} \varphi(gf) &= \sum_{i=1}^r x_i \otimes (gf)({x_i}^{-1}) \\ &= \sum_{i=1}^r x_i \otimes f({x_i}^{-1} g) \end{aligned}$$

but writing ${x_i}^{-1} g = h_i {x_{i'}}^{-1}$ for some $h_i \in H$ and possibly different coset representative $x_{i'}$, we get

$$\begin{aligned} &= \sum_{i=1}^r x_i \otimes f(h_i {x_{i'}}^{-1}) \\ &= \sum_{i=1}^r x_i \otimes h_i f({x_{i'}}^{-1}) \\ &= \sum_{i=1}^r x_i h_i \otimes f({x_{i'}}^{-1}) \\ &= \sum_{i=1}^r g x_{i'} \otimes f({x_{i'}}^{-1}) \\ &= g\varphi(f). \end{aligned}$$

Next we check that φ is injective. If $\varphi(f) = 0$, then $f({x_i}^{-1}) = 0$ and hence $f(h{x_i}^{-1}) = hf({x_i}^{-1}) = 0$ for any $h \in H$ and $i = 1, \ldots, r$. However since $G = \bigcup H{x_i}^{-1}$ we see that f is identically 0 on all of G, i.e. $f \equiv 0$.

Solution by Bradley Froehle (bfroehle@math.berkeley.edu).

Finally we verify that φ is surjective. It suffices to construct an element $f \in \mathrm{Hom}_{kH}(kG, V)$ so that $\varphi(f) = x_i \otimes v$ for any $v \in V$ and $i = 1, \ldots, r$. Let us define f on elements $g \in G$ as

$$f(g) = \begin{cases} hv & \text{if } g = h{x_i}^{-1} \ (h \in H) \\ 0 & \text{else} \end{cases}$$

and extend linearly to kG. The $\varphi(f) = x_i \otimes v$ and hence φ is surjective.

We conclude by noting that since φ is a kG-module isomorphism, U is kG-isomorphic to V^G. □

Remark. *Note that this proof relies heavily on the fact that $[G : H] < \infty$, and in general the conclusion does not hold if $[G : H] = \infty$.*