Math 252: Representation Theory

Exercises XVI

Problem 9. *Let $G = \mathscr{D}_r$ be the dihedral group of order $n = 2r$ and write $r = 2m+1$ if r is odd, and $r = 2m$ if r is even. Show that the counting formula for the number t of involutions in G ($t = \sum_{\chi \neq 1} s(\chi)\chi(1)$) leads to $t = 2m+1$. (Check that this is the correct number by an explicit enumeration of the involutions.) Referring to the bound $t^2 \leq (s-1)(n-1)$ (where $s = |\operatorname{Irr}(G)|$), show that the difference $(s-1)(n-1) - t^2$ is m when r is odd, and $3(m-1)$ when r is even.*

Solution. Recall that in Exercise IV.10, we showed that all (complex) irreducible representations of $\mathscr{D}_r$ are realizable over $\mathbb{R}$. Therefore $s(\chi) = 1$ for all $\chi \in \operatorname{Irr}(G)$.

We proceed with a separate analysis depending on whether r is odd or even.

- $r = 2m+1$ is odd:

  Recall from Exercise IV.10 that the dimensions of the irreducible representations of $\mathscr{D}_{2m+1}$ are

  $$\{1, 1, \underbrace{2, 2, \ldots, 2}_{m \text{ times}}\}.$$

  Thus $t = \sum_{\chi \neq 1} s(\chi)\chi(1) = 2m+1$. This agrees with what we would expect, as $\mathscr{D}_{2m+1}$ contains $2m+1$ reflections and no rotations of order 2.

  The overestimation in the bound $t^2 \leq (s-1)(n-1)$ is given by

  $$(s-1)(n-1) - t^2 = (m+2-1)(4m+2-1) - (2m+1)^2 = m.$$

- $r = 2m$ is even:

  Recall that the dimensions of the irreducible representations of $\mathscr{D}_{2m}$ are

  $$\{1, 1, 1, 1, \underbrace{2, 2, \ldots, 2}_{m-1 \text{ times}}\}.$$

  Thus $t = \sum_{\chi \neq 1} s(\chi)\chi(1) = 3 + 2(m-1) = 2m+1$. This is expected as $\mathscr{D}_{2m}$ contains $2m$ reflections and one rotation of order 2.

  Lastly, we see that the overestimation in the bound $t^2 \leq (s-1)(n-1)$ is given by

  $$(s-1)(n-1) - t^2 = (m+3-1)(4m-1) - (2m+1)^2 = 3(m-1).$$

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